STEP II, 2017, Q12 MS

Part (i) required thinking about the different ways in which the total number of fish caught could be n – for each value of X, there is a corresponding value of Y. This leads to a sum. Each probability can be written using the formula for the Poisson distribution. It is useful to have an idea of what you are trying to get to (a $\mathrm{Po}(\lambda + \mu)$ distribution). Pulling out some common factors leaves something very close to a binomial expansion of $(\lambda + \mu)^n$. Artificially pulling out another factor of $\frac{1}{n!}$ leaves exactly the required expansion.

Part (ii) starts by turning the situation described into a probability statement, then using the formula for conditional probability. Substituting in the expressions from the Poisson distributions and a little algebra leads to a standard binomial expression.

Part (iii) is all about linking with part (ii). When the first fish is caught the total number of fish caught is one, and you want to know the probability that Adam caught it.

Part (iv) requires some quite subtle thinking. The expected waiting time can be split into the expected waiting time with Adam first or with Eve first. Some careful thought is required to realise that, for example, the waiting time with Adam first can be broken down into the time for a fish to be caught followed by the time for Eve to catch a fish.

(i) $$P(X+Y=n)=\sum_{r=0}^{n} \mathrm{P(X=r)(P(Y=n-r)}$$	**B2**	
$$=\sum_{r=0}^{n}\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{n-r}}{(n-r)!}$$	**B1**	
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\frac{n!}{r!\,(n-r)!}\lambda^{r}\mu^{n-r}$$	**M1**	Attempting to manipulate factorials towards a binomial coefficient
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{r}\mu^{n-r}$$	**B1**	Identifying correct binomial coefficient
$$=\frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^{n}$$	**B1**	
Which is the the formula for $Po(\lambda+\mu)$	**E1**	Recognising result. Must state parameters
	[7]	
(ii)		
$$P(X=r\mid X+Y=k)=\frac{P(X=r)\times P(Y=k-r)}{P(X+Y=k)}$$	**M2**	(may be implied by following line)
$$=\frac{\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}}{\frac{e^{-(\lambda+\mu)}}{k!}(\lambda+\mu)^{k}}$$	**A1**	
$$=\frac{k!}{r!\,(k-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^{r}\left(\frac{\mu}{\lambda+\mu}\right)^{k-r}$$	**A1**	
Which is a $B\left(k,\frac{\lambda}{\lambda+\mu}\right)$ distribution.	**E1**	Parameters must be stated.
	[5]	
(iii) This corresponds to r=1, k=1 from (ii)	**M2**	Can be implied by correct answer.
So probability is $\frac{\lambda}{\lambda+\mu}$.	**A1**	
(iv)	**[3]**	
Expected waiting time given that Adam is first is waiting time for first fish plus waiting time for Eve $\left(=\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)$	**B2**	Also accept waiting time given Eve is first. Must be clearly identified.
Expected waiting time is: E(Waiting time|Adam first)P(Adam first)+E(Waiting time|Eve first)P(Eve first)	**M2**	
$=\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)\times\frac{\lambda}{\lambda+\mu}+\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\times\frac{\mu}{\lambda+\mu}$	**A1**	
$$=\frac{1}{\lambda}+\frac{1}{\mu}-\frac{1}{\lambda+\mu}$$		No need for this algebraic simplification.
	[5]	