STEP II, 2017, Q1

1 **Note:** In this question you may use without proof the result $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \dfrac{1}{1+x^2}$.

Let

$$I_n = \int_0^1 x^n \arctan x \, \mathrm{d}x \, ,$$

where $n = 0, 1, 2, 3, \ldots$.

(i) Show that, for $n \geqslant 0$,

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} \, \mathrm{d}x$$

and evaluate I_0.

(ii) Find an expression, in terms of n, for $(n+3)I_{n+2} + (n+1)I_n$.
Use this result to evaluate I_4.

(iii) Prove by induction that, for $n \geqslant 1$,

$$(4n+1)I_{4n} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r \frac{1}{r} \, ,$$

where A is a constant to be determined.