# STEP II, 2016, Q8 MS

**Question 8**

The integral required at the start of the question should be a straightforward one to evaluate. When making a sketch to illustrate the result in the second part, ensure that the sum is indicated by a series of rectangles, with the graph of the curve passing through the midpoints of the tops.

In part (i), the integral that would match the sum given results in an answer of 2, so this is the first of the estimates. The remaining estimates arise from using the integral to estimate most of the sum, but taking the first few terms as the exact values (so in each case the integration is taken from a different lower limit).

For part (ii), evaluate the integral for one particular term of the sum and note that it is approximately $\frac{1}{4r^4}$. Finally, using the most accurate estimate for $E\left(\frac{33}{20}\right)$ the sum from $r = 3$ onwards can be calculated and then the first two values of $\frac{1}{r^4}$ can be added to achieve the desired result.

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|---|---|---|
| | $$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{m-\frac{1}{2}}^{\infty} = \frac{2}{2m-1}$$ | M1 A1 |
| | Sketch of $y = \frac{1}{x^2}$ | B1 |
| | Rectangle drawn with height $\frac{1}{m^2}$ and width going from $m - \frac{1}{2}$ to $m + \frac{1}{2}$ | B1 |
| | Rectangle drawn with height $\frac{1}{n^2}$ and width going from $n - \frac{1}{2}$ to $n + \frac{1}{2}$ | B1 |
| | At least one other rectangle in between, showing that no gaps are left between the rectangles. | B1 |
| | An explanation that the rectangle areas match the sum. | B1 |
| | | |
| (i) | Taking $m = 1$ and a very large value of $n$, the approximations for $E$ is $2 - \frac{2}{2n+1}$ | M1 |
| | Therefore with $m = 1$, $E \to 2$ as $n \to \infty$ | A1 |
| | If $m = 2$, $E \to \frac{2}{3}$ as $n \to \infty$ | M1 |
| | Therefore an approximation for $E$ is $1 + \int_{\frac{3}{2}}^{\infty} \frac{1}{x^2}dx = \frac{5}{3}$ | A1 |
| | Similarly, if $m = 3$, $E \to \frac{2}{5}$ as $n \to \infty$ | |
| | Therefore an approximation for $E$ is $1 + \frac{1}{4} + \int_{\frac{5}{2}}^{\infty} \frac{1}{x^2}dx = \frac{5}{4} + \frac{2}{5} = \frac{33}{20}$ | A1 |
| | | |
| (ii) | $$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{r-\frac{1}{2}}^{r+\frac{1}{2}} = \frac{2}{2r-1} - \frac{2}{2r+1} = \frac{4}{4r^2-1}$$ | M1 A1 |
| | The error is $\frac{4}{4r^2-1} - \frac{1}{r^2} = \frac{1}{(4r^2-1)r^2} \approx \frac{1}{4r^4}$ for large values of $r$. | M1 A1 |
| | | |
| | The error in the estimate for $E$ is approximately $$\sum_{r=1}^{\infty} \frac{1}{r^4}$$ | B1 |
| | Using $E \approx \frac{33}{20}$, $$\sum_{r=3}^{\infty} \frac{1}{4r^4} \approx \frac{33}{20} - 1.645 = 0.005$$ | M1 |
| | Therefore: $$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1 + 0.0625 + 4(0.005) = 1.083$$ | M1 A1 |