STEP II, 2016, Q7 MS

Question 7

The first result can be shown by using a substitution into the integral, being careful to explain the change of sign when the limits of the integral are switched.

Simple application of knowledge of trigonometric graphs once the substitution has been made can be used to show that twice the integral is equivalent to integrating the function 1 over the interval.

Similarly, the remaining integrals can all be rearranged using standard trigonometric identities and knowledge of logarithms into forms that can be integrated from standard results once the substitution from (*) has been made.

	Let $y = a - x$:	
	Limits: $x = a \;\rightarrow y = 0$ $x = 0 \;\rightarrow y = a$	B1
	$\frac{dy}{dx} = -1$	B1
	$\int_0^a f(x)\,dx = -\int_a^0 f(a-y)\,dy$	
	Swapping limits of the integral changes the sign (and we can replace y by x in the integral on the right: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$	B1
(i)	Using (*): $\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}dx = \int_0^{\frac{1}{2}\pi} \frac{\sin(\frac{1}{2}\pi - x)}{\cos(\frac{1}{2}\pi - x) + \sin(\frac{1}{2}\pi - x)}dx$	M1
	$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}dx = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \sin x}dx$	A1
	Therefore $2\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}dx = \int_0^{\frac{1}{2}\pi} \frac{\sin x + \cos x}{\cos x + \sin x}dx$ $= \int_0^{\frac{1}{2}\pi} 1 dx$ $= \frac{1}{2}\pi$	M1 A1
	$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}dx = \frac{1}{4}\pi$	A1
(ii)	Using (*): $\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}dx = \int_0^{\frac{1}{4}\pi} \frac{\sin(\frac{1}{4}\pi - x)}{\cos(\frac{1}{4}\pi - x) + \sin(\frac{1}{4}\pi - x)}dx$	
	$\frac{\sin\left(\frac{1}{4}\pi - x\right)}{\cos\left(\frac{1}{4}\pi - x\right) + \sin\left(\frac{1}{4}\pi - x\right)} = \frac{\frac{\sqrt{2}}{2}(\cos x - \sin x)}{\frac{\sqrt{2}}{2}(\cos x + \sin x + \cos x - \sin x)}$ $= \frac{1}{2}(1 - \tan x)$	M1 M1 A1
	$\frac{1}{2}\int_0^{\frac{1}{4}\pi} 1 - \tan x\,dx = \frac{1}{2}[x - \ln\lvert\sec x\rvert]_0^{\frac{1}{4}\pi}$	M1
	$= \frac{1}{8}\pi - \frac{1}{4}\ln 2$	A1

(iii)	Using (*): $$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx = \int_0^{\frac{1}{4}\pi} \ln\left(1+\tan\left(\frac{1}{4}\pi - x\right)\right)dx$$ $$= \int_0^{\frac{1}{4}\pi} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$ $$= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{2}{1+\tan x}\right)dx$$	**M1**
	Therefore, if $I = \int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx$, then $I = \frac{1}{4}\pi\ln 2 - I$	**M1**
	$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx = \frac{1}{8}\pi\ln 2$$	**A1**
(iv)	Using (*): $$I = \int_0^{\frac{1}{4}\pi} \frac{x}{\cos x\,(\cos x+\sin x)}dx = \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi - x}{\frac{\sqrt{2}}{2}(\cos x+\sin x)\sqrt{2}\cos x}dx$$ $$= \frac{1}{4}\pi\int_0^{\frac{1}{4}\pi} \frac{1}{(\cos x+\sin x)\cos x}dx - I$$	**M1**
	$$\int_0^{\frac{1}{4}\pi} \frac{1}{(\cos x+\sin x)\cos x}dx = \int_0^{\frac{1}{4}\pi} \frac{\sec^2 x}{1+\tan x}dx = [\ln(1+\tan x)]_0^{\frac{1}{4}\pi}$$	**M1** **A1**
	Therefore $$2I = \frac{1}{4}\pi\ln 2$$ $$I = \frac{1}{8}\pi\ln 2$$	**A1**