STEP II, 2016, Q6 MS

Question 6

Parts (i) and (ii) only require verification in each of the cases, so simply differentiate the functions given and substitute into the differential equation to confirm that they are solutions. Remember to check as well that the boundary conditions are satisfied.

For part (iii), differentiate the given formula for z and substitute into the differential equation. By observing that the new differential equation is of the same form as (*), but for $2n$ instead of n, the expression for $y_{2n}(x)$ can be established.

For part (iv), again differentiate the given formula, being careful about the application of the chain rule and substitute. Again, by comparing with (*) the final result should be clear.

(i)	$$(1-x^2)\left(\frac{dy}{dx}\right)^2+y^2=1$$	
	If $y=x$, then $\frac{dy}{dx}=1$ and so LHS becomes $$(1-x^2)(1)^2+(x)^2=1=RHS$$	B1
	$y_1(1)=1$, so the boundary condition is also satisfied.	B1
(ii)	$$(1-x^2)\left(\frac{dy}{dx}\right)^2+4y^2=4$$	
	If $y=2x^2-1$, then $\frac{dy}{dx}=4x$ and so LHS becomes $$(1-x^2)(4x)^2+4(2x^2-1)^2=16x^2-16x^4+4(4x^4-4x^2+1)$$ $$=4=RHS$$	M1 A1
	$y_2(1)=2(1)^2-1=1$, so the boundary condition is also satisfied.	B1
(iii)	If $z(x)=2\big(y_n(x)\big)^2-1$, then $\frac{dz}{dx}=4y_n(x)\frac{dy_n}{dx}$	M1 A1
	Substituting in to the LHS of the differential equation: $$(1-x^2)\left(4y_n\frac{dy_n}{dx}\right)^2+4n^2(2(y_n)^2-1)^2$$	M1
	$$=16y_n^2\left[(1-x^2)\left(\frac{dy_n}{dx}\right)^2+n^2y_n^2-n^2\right]+4n^2$$	M1 A1
	Since y_n is a solution of (*) when $k=n$: $$=4n^2$$	A1
	Since $z(1)=2(1)^2-1=1$, z is a solution to (*) when $k=2n$.	M1
	Therefore $y_{2n}(x)=2\big(y_n(x)\big)^2-1$	A1
(iv)	$$\frac{dv}{dx}=\frac{dy_n}{dx}\big(y_m(x)\big)\frac{dy_m}{dx}(x)$$	B1
	Substituting into LHS of (*) with $k=mn$: $$(1-x^2)\left(\frac{dy_n}{dx}\big(y_m(x)\big)\frac{dy_m}{dx}(x)\right)^2+(mn)^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	M1
	$$=\frac{dy_n}{dx}\big(y_m(x)\big)\left((1-x^2)\left(\frac{dy_m}{dx}(x)\right)^2\right)+m^2n^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	M1
	From (*), $(1-x^2)\left(\frac{dy_m}{dx}(x)\right)^2=m^2-m^2y_m(x)^2$	M1
	Therefore, we have: $$\frac{dy_n}{dx}\big(y_m(x)\big)(m^2-m^2y_m(x)^2)+m^2n^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	
	Let $u=y_m(x)$, then this simplifies to $$m^2[(1-u^2)\frac{dy_n}{dx}(u)+n^2y_n(u)^2]$$	M1
	And by applying (*) when $k=n$, this simplifies to m^2n^2, so v satisfies (*) when $k=mn$.	A1
	$v(1)=y_n\big(y_m(1)\big)=y_n(1)=1$, so $v(x)=y_{mn}(x)$	A1