STEP II, 2016, Q6

6 This question concerns solutions of the differential equation

$$(1-x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + k^2y^2 = k^2 \qquad (*)$$

where k is a positive integer.

For each value of k, let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k.

(i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.

(ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2\big(y_n(x)\big)^2 - 1$. Show that

$$(1-x^2)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + 4n^2z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

(iv) Let $v(x) = y_n\big(y_m(x)\big)$. Show that $v(x) = y_{mn}(x)$.