STEP II, 2016, Q5 MS

Question 5

The binomial expansion for $(1-x)^{-N}$ should be easy enough, it is then required to write the product in terms of factorials so that the expression can be written in terms of $\binom{p}{q}$.

Since the expansion of $(1-x)^{-1}$ involves a coefficient of 1 for every term, the coefficient of x^n in the expansion of $(1-x)^{-1}(1-x)^{-N}$ is simply the sum of the coefficients of all of the terms in the expansion of $(1-x)^{-N}$ up to and including the term in x^n.

The products in the sum on the right-hand side of the result in part (ii) should be recognisable as binomial coefficients in the case where the power is a positive integer, so use

$$(1+x)^p(1+x)^q \equiv (1+x)^{p+q}$$

and compare coefficients as in part (i).

Similarly for part (iii), identify that the result will come from consideration of

$$(1+x)^{N+m}(1+x)^{-m} \equiv (1+x)^N.$$

(i)	Coefficient of x^n is $\frac{-N(-N-1)\dots(-N-n+1)}{n!}(-1)^n = \frac{N(N+1)\dots(N+n-1)}{n!}$ or $\binom{N+n-1}{N-1}$ or $\binom{N+n-1}{n}$	M1 M1 A1
	Expansion is therefore: $\sum_{r=0}^{\infty}\frac{N(N+1)\dots(N+r-1)}{r!}x^r$ or $\sum_{r=0}^{\infty}\binom{N+r-1}{N-1}x^r$	B1
	$(1-x)^{-1} = 1 + x + x^2 + \cdots$	B1
	Therefore the coefficient of x^n in the expansion of $(1-x)^{-1}(1-x)^{-N}$ is the sum of the coefficients of the terms up to x^n in the expansion of $(1-x)^{-N}$. $$\sum_{j=0}^{n}\binom{N+j-1}{j} = \binom{(N+1)+n-1}{n} = \binom{N+n}{n} \qquad (*)$$	M1 A1
(ii)	Write $(1+x)^{m+n}$ as $(1+x)^m(1+x)^n$.	B1
	When multiplying the two expansions, terms in x^r will be obtained by multiplying the term in x^j from one expansion by the term in x^{r-j} in the other expansion.	M1
	The coefficient of x^r in the expansion of $(1+x)^{m+n}$ is $\binom{m+n}{r}$	M1
	The coefficient of x^j in the expansion of $(1+x)^m$ is $\binom{m}{j}$	M1
	The coefficient of x^{r-j} in the expansion of $(1+x)^n$ is $\binom{n}{r-j}$	M1
	Therefore, summing over all possibilities: $$\binom{m+n}{r} = \sum_{j=0}^{n}\binom{m}{j}\binom{n}{r-j} \qquad (*)$$	A1
(iii)	Write $(1-x)^N$ as $(1-x)^{N+m}(1-x)^{-m}$	B1
	The coefficient of x^n in the expansion of $(1-x)^N$ is $(-1)^n\binom{N}{n}$	M1
	The coefficient of x^{n-j} in $(1-x)^{N+m}$ is $\binom{N+m}{n-j}(-1)^{n-j}$	M1 A1
	The coefficient of x^j in $(1-x)^{-m}$ is $\binom{m+j-1}{j}$	M1
	Therefore $$\sum_{j=0}^{n}\binom{N+m}{n-j}(-1)^{n-j}\binom{m+j-1}{j} = (-1)^n\binom{N}{n}$$	M1
	And so, $$\sum_{j=0}^{n}\binom{N+m}{n-j}(-1)^{j}\binom{m+j-1}{j} = \binom{N}{n} \qquad (*)$$	A1