STEP II, 2016, Q5

5 In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!} & \text{if } p \geqslant q \geqslant 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n}\,.$$

(ii) Show that, for any positive integers m, n and r with $r \leqslant m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}\,.$$

(iii) Show that, for any positive integers m and N,

$$\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}.$$