STEP II, 2016, Q4 MS

Question 4

The equation given can be rewritten as a quadratic in x. The discriminant then establishes the required result. To show the second result, show that $y^2 + 1 \geq (y\cos\theta - \sin\theta)^2$, which can be shown by writing $y\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ and then this result is a quadratic inequality that leads directly to the next result.

In the case $y = \frac{4+\sqrt{7}}{3}$, careful manipulation of surds shows the required result and so the value of θ must be the value of α obtained in the previous section. Finally, the value of x can be obtained by returning to the original equation and substituting in the values that are known.

(i)	$y\cos\theta-\sin\theta=\dfrac{(x^2+x\sin\theta+1)\cos\theta-(x^2+x\cos\theta+1)\sin\theta}{x^2+x\cos\theta+1}$ $=\dfrac{(x^2+1)(\cos\theta-\sin\theta)}{x^2+x\cos\theta+1}$	M1 A1
	$y-1=\dfrac{x(\sin\theta-\cos\theta)}{x^2+x\cos\theta+1}$	B1
	$(y\cos\theta-\sin\theta)^2=\dfrac{(x^2+1)^2(\sin\theta-\cos\theta)^2}{(x^2+x\cos\theta+1)^2}$ $=(y-1)^2\times\dfrac{(x^2+1)^2}{x^2}$	M1
	$\dfrac{(x^2+1)^2}{x^2}=\left(x+\dfrac{1}{x}\right)^2$	M1
	Minimum value of $\left(x+\frac{1}{x}\right)^2$ is 4, therefore $(y\cos\theta-\sin\theta)^2\geq 4(y-1)^2$ $(*)$	M1 A1
	$y\cos\theta-\sin\theta$ can be written as $\sqrt{y^2+1}\cos(\theta+\alpha)$ for some value of α.	M1 A1
	Therefore $y^2+1\geq(y\cos\theta-\sin\theta)^2\geq 4(y-1)^2$	A1
	$y^2+1\geq 4y^2-8y+4$	
	$3y^2-8y+3\leq 0$	M1
	Critical values are: $y=\frac{8\pm\sqrt{(8)^2-4(3)(3)}}{2(3)}$ $\dfrac{4-\sqrt{7}}{3}\leq y\leq\dfrac{4+\sqrt{7}}{3}$	A1
(ii)	If $y=\frac{4+\sqrt{7}}{3}$, then $\sqrt{y^2+1}=\sqrt{\frac{16+8\sqrt{7}+7}{9}+1}=\sqrt{\frac{32+8\sqrt{7}}{9}}$	
	$2(y-1)=\frac{2+2\sqrt{7}}{3}$	M1
	$\left(\frac{2+2\sqrt{7}}{3}\right)^2=\frac{4+8\sqrt{7}+28}{9}$, so $\sqrt{y^2+1}=2(y-1)$	A1
	Since $\sqrt{y^2+1}=2(y-1)$, the value of θ must be the value of α when $y\cos\theta-\sin\theta$ is written as $\sqrt{y^2+1}\cos(\theta+\alpha)$.	B1
	Therefore $\tan\theta=\frac{1}{y}=\frac{4-\sqrt{7}}{3}$	M1 A1
	To find x: $x^2y+xy\cos\theta+y=x^2+x\sin\theta+1$	M1
	$x^2(y-1)+x(y\cos\theta-\sin\theta)+y-1=0$	
	Since $y\cos\theta-\sin\theta=\pm 2(y-1)$, and $y-1\neq 0$ this simplifies to: $x^2\pm 2x+1=0$	M1
	So we have either $x=1$ or $x=-1$	A1