STEP II, 2016, Q3 MS

Question 3

The differentiation to show the result in part (i) should not present much difficulty, although it is important to show that all of the terms (and no others) are present.

For part (ii) observe that each individual term of $f_n(x)$ has a positive coefficient, so for any positive value of x the value of $f_n(x)$ must be positive.

For part (iii), use the result in part (i) to rewrite $f_n{}'(x)$ in terms of $f_n(x)$ and note that $f_n(a)$ and $f_n(b)$ must be 0. This means that any pair of roots must have a gradient of the same sign, which leads to an argument that there must be another root between the two. As this would lead to an infinite number of roots to a polynomial, there cannot be more than one root.

To establish the number of roots in the two cases consider the behaviour of the graph as $x \to \infty$ and as $x \to -\infty$

(i)	$$f_n'(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$	B1
(ii)	If a is a root of the equation then $f_n(a) = 0$	B1
	Each of the terms of $f(a)$ will be positive if $a > 0$.	M1
	Therefore $f_n(a) > 0$	A1
(iii)	$f_n'(a) = f_n(a) - \frac{a^n}{n!} = -\frac{a^n}{n!}$, and similarly for b.	M1 A1
	Since a and b are both negative, $f_n'(a)$ and $f_n{}'(b)$ must have the same sign.	M1 M1
	Therefore $f_n'(a)f_n'(b) > 0$	A1
	Two cases (positive and negative gradients) Sketch needed for each	B1 B1
	Since the graph is continuous, there must be an additional root between a and b.	M1 A1
	This would imply infinitely many roots.	M1
	But $f_n(x)$ is a polynomial of degree n, so has at most n roots	M1
	Therefore there is at most one root.	A1
	If n is odd then $f_n(x) \to -\infty$ as $x \to -\infty$ and $f_n(x) \to \infty$ as $x \to \infty$ There is one real root.	M1 A1
	If n is even then $f_n(x) \to \infty$ as $x \to -\infty$ and $f_n(x) \to \infty$ as $x \to \infty$ There are no real roots.	M1 A1