STEP II, 2016, Q3

3 For each non-negative integer n, the polynomial f_n is defined by

$$\mathrm{f}_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\,.$$

(i) Show that $\mathrm{f}_n'(x) = \mathrm{f}_{n-1}(x)$ (for $n \geqslant 1$).

(ii) Show that, if a is a real root of the equation

$$\mathrm{f}_n(x) = 0\,, \qquad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geqslant 2$. Show that $\mathrm{f}_n'(a)\,\mathrm{f}_n'(b) > 0$ and use a sketch to deduce that $\mathrm{f}_n(c) = 0$ for some number c between a and b.

Deduce that $(*)$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?