STEP II, 2016, Q2 MS

Question 2

Substitute $c = a + b$ into the expression to show that $a + b - c$ is a factor. Once this is done, the symmetry shows that $b + c - a$ and $c + a - b$ must also be factors and therefore there is just a constant multiplier that needs to be deduced to obtain the full factorisation of (*).

For part (i), choices of a, b and c need to be made so that

$$a + b + c = x + 1$$

$$a^2 + b^2 + c^2 = \frac{2x^2 + 5}{2}$$

$$a^3 + b^3 + c^3 = \frac{4x^3 + 13}{4}$$

Once these have been identified the solutions to the equation follow from the factorisation already deduced.

Once the substitution $d + e = c$ has been made it is only necessary to identify the parts of the expression which differ from (*) in the first part of the question (which arise from the c^2 and c^3 terms). The factorisation and solution of the equation then follow a similar process to the first part of the question.

	Let $c = a + b$: $$(2a+2b)^3 - 6(2a+2b)(a^2+b^2+(a+b)^2) + 8(a^3+b^3+(a+b)^3)$$	M1
	$= 8(a+b)^3 - 24(a+b)(a^2+ab+b^2) + 8(2a^3+3a^2b+3ab^2+2b^3)$	
	$= 0$	M1
	Therefore $(a+b-c)$ is a factor of (*)	A1
	By symmetry, $(b+c-a)$ and $(c+a-b)$ must also be factors.	B1
	So (*) must factorise to $k(a+b-c)(b+c-a)(c+a-b)$	M1
	To obtain the correct coefficient of a^3, $k = -3$	M1
	(*) factorises to $-3(a+b-c)(b+c-a)(c+a-b)$	A1
(i)	To match the equation given, we need $a+b+c = x+1$, $a^2+b^2+c^2 = \frac{5}{2}$ and $a^3+b^3+c^3 = \frac{13}{4}$.	M1
	$a = x, b = \frac{3}{2}, c = -\frac{1}{2}$	A1
	The equation therefore factorises to $$-3(x+2)(1-x)(x-2) = 0$$	M1
	$x = -2, 1 \text{ or } 2$	A1
(ii)	Let $d + e = c$ in (*): $a+b-d-e$ is a factor of $(a+b+d+e)^2 - 6(a+b+d+e)(a^2+b^2+(d+e)^2) + 8(a^3+b^3+(d+e)^3)$	
	Which is: $$(a+b+d+e)^2 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3)$$ $$-6(a+b+d+e)(2de) + 8(3d^2e+3de^2)$$	M1
	$-6(a+b+d+e)(2de) + 8(3d^2e+3de^2) = -12ade - 12bde + 12d^2e + 12de^2$	M1
	Which is $-12(a+b-d-e)(de)$. Therefore $(a+b-d-e)$ is a factor of: $$(a+b+d+e)^2 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3)$$	A1
	By symmetry, $a-b-d+e$ and $a-b+d-e$ must also be factors, so it must factorise to: $$k(a+b-d-e)(a-b-c+d)(a-b+c-d)$$	M1
	To obtain the correct coefficient we require $k = 3$.	A1
	To match the equation given we need $a+b+c+d = x+6$, $a^2+b^2+c^2+d^2 = x^2+14$ and $a^3+b^3+c^3+d^3 = x^3+36$	M1
	$a = x, b = 1, c = 2, d = 3$	A1
	The equation therefore factorises to $$3x(x-4)(x-2)$$	M1
	$x = 0, 2 \text{ or } 4$	A1