# STEP II, 2016, Q1 MS

**Question 1**

Use $t_1$ and $t_2$ to represent the value of the parameter $t$ at each of the points P and Q. The equations of the two tangents can therefore be found in terms of $t_1$ and $t_2$ and the fact that POQ is a right angle can be used to find a relationship between $t_1$ and $t_2$. The point of intersection of the two tangents can therefore be found in terms of just $t_1$ and this is a pair of parametric equations for the curve that the point of intersection makes.

Substituting the parametric equations for $C_1$ into the equation for $C_2$ gives a cubic equation in $t^2$ which can be solved by inspection to show that there are just two intersections and so the two curves just touch, but do not cross.

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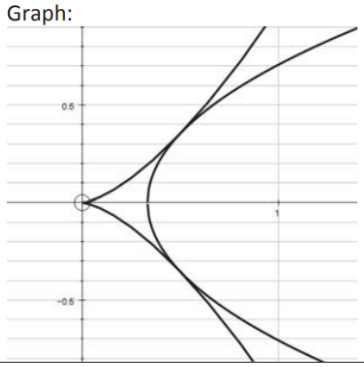
|---|---|---|
| | If the value of the parameter at $P$ is $p$ and the value of the parameter at $Q$ is $q$: | |
| | Gradient of line $OP$ is $\frac{p^3-0}{p^2-0} = p$ and similarly the gradient of $OQ$ is $q$. | M1 |
| | If the angle at $O$ is a right angle, then $pq = -1$ | A1 |
| | | |
| | $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$ | M1 |
| | Therefore $\frac{dy}{dx} = \frac{3}{2}t$ | A1 |
| | Equation of tangent at the point $(t^2, t^3)$: $$y - t^3 = \frac{3}{2}t(x - t^2)$$ | M1 A1 |
| | $$\frac{3}{2}p(x - p^2) + p^3 = \frac{3}{2}q(x - q^2) + q^3$$ | M1 |
| | $$3px - 3p^3 + 2p^3 = 3qx - 3q^3 + 2q^3$$ | |
| | $$x = \frac{p^3 - q^3}{3(p - q)} = \frac{1}{3}(p^2 + pq + q^2)$$ | A1 |
| | Substitute for $y$: $$y - p^3 = \frac{3}{2}p\left(\frac{1}{3}(p^2 + pq + q^2) - p^2\right) + p^3$$ | M1 |
| | $$y = \frac{1}{2}pq(p + q)$$ | A1 |
| | | |
| | Use $pq = -1$: $$x = \frac{p^4 - p^2 + 1}{3p^2}$$ | |
| | $$y = -\frac{p^2 - 1}{2p}$$ | B1 |
| | | |
| | $$4y^2 = \frac{p^4 - 2p^2 + 1}{p^2} = 3x - 1 \qquad (*)$$ | M1 A1 |
| | | |
| | If $C_1$ and $C_2$ meet then there must be a value of $t$ such that: $$4t^6 = 3t^2 - 1$$ | B1 |
| | $$4t^6 - 3t^2 + 1 = 0$$ | |
| | $$(2t^2 - 1)(2t^4 + t^2 - 1) = 0$$ | M1 |
| | $$(2t^2 - 1)^2(t^2 + 1) = 0$$ | A1 |
| | Therefore, points of intersection only when $t = \pm\frac{\sqrt{2}}{2}$ | B1 |
| | Graph:  | B1 B1 B1 |