STEP II, 2016, Q12 MS

Question 12

Replace B with $(B \cup C)$ in the result that you must start with and then observe that $A \cap (B \cup C)$ is the same as $(A \cap B) \cup (A \cap C)$. The corresponding result for four events should be clear, but care must be taken to include all of the possible pairs.

The results for parts (i), (ii) and (iii) should be clear from consideration of arrangements in each case and the result required follows from the generalisation of the result from the start of the question.

The probability that the first card is in the correct position and none of the others is can be established and therefore the probability that exactly one card is in the correct position will be n times that.

	$P(A\cup B\cup C)=P((A\cup B)\cup C)=P(A\cup B)+P(C)-P((A\cup B)\cap C)$	M1
	$P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C))$ $=P(A\cap C)+P(B\cap C)-P((A\cap C)\cap(B\cap C))$	M1
	$P((A\cap C)\cap(B\cap C))=P(A\cap B\cap C)$	M1
	Therefore: $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$	A1
	$P(A\cup B\cup C\cup D)=P(A)+P(B)+P(C)+P(D)$ $-P(A\cap B)-P(A\cap C)-P(A\cap D)$ $-P(B\cap C)-P(B\cap D)-P(C\cap D)$ $+P(A\cap B\cap C)+P(A\cap B\cap D)+P(A\cap C\cap D)+P(B\cap C\cap D)$ $-P(A\cap B\cap C\cap D)$	B1 B1
(i)	$P(E_i)=\frac{1}{n}$	B1
(ii)	There are a total of $n!$ arrangements possible.	M1
	$(n-2)!$ of these will have the ith and jth in the correct position.	M1
	$P(E_i\cap E_j)=\frac{1}{n(n-1)}$	A1
(iii)	By similar reasoning to (ii) the probability will be $\frac{1}{n(n-1)(n-2)}$	M1
		M1
		A1
	At least one card is in the position as the number it bears is the union of all of the E_is	B1
	$P\left(\bigcup_{1\le i\le n}E_i\right)=\sum_{1\le i\le n}P(E_i)-\sum_{1\le i<j\le n}P(E_i\cap E_j)+\sum_{1\le i<j<k\le n}P(E_i\cap E_j\cap E_k)-\cdots$ $+(-1)^{n+1}P(E_1\cap E_2\cap\ldots\cap E_n)$	M1
	$P\left(\bigcup_{1\le i\le n}E_i\right)=n\times\frac{1}{n}-\binom{n}{2}\times\frac{1}{n(n-1)}+\binom{n}{3}\times\frac{1}{n(n-1)(n-2)}-\cdots$ $+(-1)^{n+1}\times\frac{1}{n(n-1)(n-2)\ldots 2\times 1}$	M1 M1
	$P\left(\bigcup_{1\le i\le n}E_i\right)=1-\frac{1}{2!}+\frac{1}{3!}-\cdots+(-1)^{n+1}\frac{1}{n!}$	A1
	The probability that no cards are in the same position as the number they bear is $\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}$	M1
	Therefore the probability that exactly one card is in the same position as the number it bears is $n\times P(E_1)\times$ the probability that no card from a set of $(n-1)$ is in the same position as the number it bears.	
	$\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n-1}\frac{1}{(n-1)!}$	A1