STEP II, 2015, Q8 MS

Question 8

The first part of the question follows from consideration of similar triangles in the diagram if the line through P and the centres of the circles is added. For the second part, expressions can be written down for the position vectors of Q and R by noting that the same method as in part (i) will still apply. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ can then be compared to show that one is a multiple of the other.

For the final part of the question, note that Q will lie halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$.

Question 8

(i)	Let $\boldsymbol{a}$ be the vector from the centre of C_2 to P.	
	Using similar triangles, the vector from the centre of C_1 to P is $\frac{r_1}{r_2}\boldsymbol{a}$.	**M1 A1**
	Therefore $\frac{r_1}{r_2}\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{x_2} - \boldsymbol{x_1}$, since these are both expressions for the vector from the centre of C_1 to the centre of C_2.	**M1**
	So $\boldsymbol{a} = \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1})$	**A1**
	The position vector of P is $\boldsymbol{x_2} + \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1}) = \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2}$	**M1 A1**
(ii)	The position vectors of Q and R will be $\frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1}$ and $\frac{r_2\boldsymbol{x_3} - r_3\boldsymbol{x_2}}{r_2 - r_3}$.	**B1**
	Therefore, $\overrightarrow{PQ} = \frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1} - \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2} = \frac{\boldsymbol{x_1}[r_3(r_1-r_2)+r_2(r_3-r_1)] - \boldsymbol{x_2}[r_1(r_3-r_1)] - \boldsymbol{x_3}[r_1(r_1-r_2)]}{(r_3-r_1)(r_1-r_2)}$	**M1 A1**
	$\overrightarrow{PQ} = \frac{r_1}{(r_3-r_1)(r_1-r_2)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1**
	Similarly, $\overrightarrow{QR} = \frac{r_3}{(r_2-r_3)(r_3-r_1)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1** **M1 A1**
	Since they are multiples of each other the points P, Q and R must lie on the same straight line.	**B1**
(iii)	Q lies halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$	**B1**
	Therefore $\frac{r_1}{(r_3-r_1)(r_1-r_2)} = \frac{r_3}{(r_2-r_3)(r_3-r_1)}$	**M1**
	So, $r_1(r_2 - r_3) = r_3(r_1 - r_2)$	
	Which simplifies to $r_1r_2 + r_2r_3 = 2r_1r_3$	**M1 A1**