STEP II, 2015, Q7 MS

Question 7

For part (i) note that the lines joining the centres of the two circles and one of the points where the bisection occurs form a right-angled triangle, so the radius of the new circle can be calculated. To show that no such circle can exist when $r < a$ note that the diametrically opposite points on C must be a distance of $2a$ apart, and no two points on a circle of radius r can be that far apart. For the case $r = a$ note that the new circle would be the same as C (and so would have more than two intersection points).

For part (ii) a similar method can be used to deduce the distances between the centre of the new circle and each of C_1 and C_2. From these distances equations can be formed relating the x and y coordinates of the centre of the new circle. It is then an easy task to eliminate the y-coordinate of the centre of the circle from the equations to get the given value of the x-coordinate.

The expression for y can easily be found by substituting back into the equations obtained from the distance between the centres of two of the circles. Once this is done, note that $y^2 \geq 0$ to obtain the final inequality.

Question 7

(i)	Most likely examples: $x^2 + \left(y \pm \sqrt{r^2 - a^2}\right)^2 = r^2$ and $\left(x \pm \sqrt{r^2 - a^2}\right)^2 + y^2 = r^2$	**M1 M1 A1**
	If $r < a$ then there cannot be two points on the circle that are a distance of $2a$ apart and any two diametrically opposite points on C must be a distance of $2a$ apart.	**B1**
	If $r = a$ then the circle must be the same as C, so there is not exactly 2 points of intersection.	**B1**
(ii)	The distances of the centre of D from the centres of C_1 and C_2 are $\sqrt{r^2 - {a_1}^2}$ and $\sqrt{r^2 - {a_2}^2}$.	**M1 A1 B1**
	If the x-coordinate of the centre of D is x, then the y-coordinate is given by $r^2 - {a_1}^2 = y^2 + (d + x)^2$ and $r^2 - {a_2}^2 = y^2 + (d - x)^2$	**B1 B1**
	Therefore, $(d + x)^2 - (d - x)^2 = (r^2 - {a_1}^2) - (r^2 - {a_2}^2)$	**M1**
	$4dx = {a_2}^2 - {a_1}^2$ and so $x = \frac{{a_2}^2 - {a_1}^2}{4d}$.	**M1 A1**
	Therefore, the y-coordinate of the centre of D satisfies $y^2 = r^2 - {a_1}^2 - \left(d + \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$ and $y^2 = r^2 - {a_2}^2 - \left(d - \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	**B1**
	So $2y^2 = 2r^2 - {a_1}^2 - {a_2}^2 - \left(d + \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2 - \left(d - \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	
	$2y^2 = 2r^2 - {a_1}^2 - {a_2}^2 - 2d^2 - 2\left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	
	So, $y = \sqrt{r^2 - \frac{{a_1}^2 + {a_2}^2}{2} - d^2 - \left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2}$	
	Therefore, $r^2 - \frac{{a_1}^2 + {a_2}^2}{2} - d^2 - \left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2 \geq 0$	**B1**
	$16r^2d^2 - 8{a_1}^2d^2 - 8{a_2}^2d^2 - 16d^4 - ({a_2}^2 - {a_1}^2)^2 \geq 0$	**M1 M1**
	$16r^2d^2 \geq 16d^4 + 8{a_1}^2d^2 + 8{a_2}^2d^2 + ({a_2}^2 - {a_1}^2)^2$	
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2)^2 + ({a_2}^2 - {a_1}^2)^2 - ({a_1}^2 + {a_2}^2)^2$	**M1**
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2)^2 - 4{a_1}^2{a_2}^2$	**M1**
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2 - 2a_1a_2)(4d^2 + {a_1}^2 + {a_2}^2 + 2a_1a_2)$	
	$16r^2d^2 \geq (4d^2 + (a_1 - a_2)^2)(4d^2 + (a_1 + a_2)^2)$	**A1**