STEP II, 2015, Q7

7 A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

(i) Let C be the circle of radius a in the x-y plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \leqslant a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\dfrac{a_2^2 - a_1^2}{4d}$.

Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \geqslant \left(4d^2 + (a_2 - a_1)^2\right)\left(4d^2 + (a_2 + a_1)^2\right).$$