STEP II, 2015, Q6 MS

Question 6

The first part of the question requires use of the $\cos(A + B)$ formula. Following this the integral should be easy to evaluate given that $\int \sec^2 x \, dx = \tan x + c$. In the second part, apply the substitution and note that the limits of the integral are reversed, which is equivalent to multiplying by -1. Following this a simple rearrangement (noting that the variable that the integration is taken over can be changed from y to x) should establish the required result. The integral at the end of this part can then be evaluated simply by applying this result along with the integral evaluated in part (i).

In the final part of the question it is tempting to make repeated applications of the result proven in part (ii). However, this is not valid as it would require the use of a function satisfying $f(\sin x) = x$, which is not possible on the interval over which the integral is defined. Instead, application of a similar substitution to part (ii) to $\int_0^\pi x^3 f(\sin x)\, dx$ will simplify to allow this integral to be evaluated based on the integration of $\frac{1}{(1+\sin x)^2}$. An application of the result from part (ii) will also be required.

Question 6

(i)	$\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$	**B1**
	$\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{1}{\left(\cos\frac{\pi}{4}\cos\frac{x}{2} + \sin\frac{\pi}{4}\sin\frac{x}{2}\right)^2}$	**B1**
	$\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{2}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$	
	$\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 \equiv \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \sin^2\frac{x}{2} = 1 + \sin x$	**M1**
	Therefore, $\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{2}{1+\sin x}$	**M1 A1**
	Therefore, $\int \frac{1}{1+\sin x} dx = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c$.	**M1 A1**
(ii)	Limits: $x = \pi \to y = 0$ $x = 0 \to y = \pi$	
	$\frac{dy}{dx} = -1$	**B1**
	$\sin(\pi - x) = \sin x$	**B1**
	Therefore, $\int_0^\pi x f(\sin x)\, dx = \int_\pi^0 (\pi - y)\, f(\sin(\pi - y))(-1) dy$	
	$\int_0^\pi x f(\sin x)\, dx = \int_0^\pi (\pi - x)\, f(\sin x) dx$	
	So, $2\int_0^\pi x f(\sin x)\, dx = \pi \int_0^\pi f(\sin x) dx$	**M1**
	$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x) dx$	**A1**
	$\int_0^\pi \frac{x}{1+\sin x} dx = \frac{\pi}{2}\int_0^\pi \frac{1}{1+\sin x} dx$, and applying the result from part (i):	
	$\int_0^\pi \frac{1}{1+\sin x} dx = \left[-\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]_0^\pi = \left(-\tan\left(-\frac{\pi}{4}\right)\right) - \left(-\tan\left(\frac{\pi}{4}\right)\right) = 2.$	**B1**
	$\int_0^\pi \frac{x}{1+\sin x} dx = \frac{\pi}{2}(2) = \pi$	**B1**
(iii)	Consider $\int_0^\pi x^3 f(\sin x)\, dx$. Making the substitution $y = \pi - x$:	
	$\int_0^\pi x^3 f(\sin x)\, dx = \int_\pi^0 (\pi - y)^3 f(\sin(\pi - y))(-1) dy$	**M1 A1**
	So, $\int_0^\pi x^3 f(\sin x)\, dx = \int_0^\pi (\pi - x)^3 f(\sin x) dx$	
	Therefore, $\int_0^\pi (2x^3 - 3\pi x^2) f(\sin x)\, dx = \int_0^\pi (\pi^3 - 3\pi^2 x)\, f(\sin x) dx$	**B1**
	$\int_0^\pi \frac{1}{(1+\sin x)^2} dx = \frac{1}{4}\int_0^\pi \sec^4\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$ $\int_0^\pi \frac{1}{(1+\sin x)^2} dx = \frac{1}{4}\int_0^\pi \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) + \tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$	**M1**
	$\int_0^\pi \frac{1}{(1+\sin x)^2} dx = \frac{1}{4}\left[-2\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{2}{3}\tan^3\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]_0^\pi = \frac{4}{3}$	**A1**
	And so, $\int_0^\pi \frac{x}{(1+\sin x)^2} dx = \frac{2\pi}{3}$	**B1**
	Therefore, $\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2} dx = \pi^3\left(\frac{4}{3}\right) - 3\pi^2\left(\frac{2\pi}{3}\right) = -\frac{2}{3}\pi^3$.	**B1**