# STEP II, 2015, Q5 MS

Question 5

The initial proof by induction is a straightforward application of the $\tan(A+B)$ formula. The final part of section (i) requires recognition that there are many possible values of $x$ to give a particular value of $\tan x$, but only one of them is the value that would be obtained by applying the arctan function. The result can therefore be shown by establishing that the difference between consecutive terms of the sequence is never more than $\pi$.

For the second part of the question a diagram of the triangle and application of the $\tan 2A$ formula shows that the value of $\alpha_n$ must be of the form used in the first part of the question. All that remains is then to show that the limit of the sum must give the required value.

**Question 5**

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|---|---|---|
| (i) | $\tan S_1 = \tan\left(\arctan\frac{1}{2}\right) = \frac{1}{1+1}$, so the formula is correct for $n=1$. | **B1** |
| | Assume that $\tan S_k = \frac{k}{k+1}$: | |
| | $S_{k+1} = S_k + \arctan\frac{1}{2(k+1)^2}$. | **M1** |
| | $\tan S_{k+1} = \frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}$ | **M1** |
| | $\tan S_{k+1} = \frac{2k(k+1)^2+(k+1)}{2(k+1)^3-k}$, which simplifies to<br>$\tan S_{k+1} = \frac{(k+1)}{(k+1)+1}$. | **M1 A1** |
| | Hence, by induction $\tan S_n = \frac{n}{n+1}$. | **A1** |
| | | |
| | Clearly, $S_1 = \arctan\left(\frac{1}{2}\right)$. | **B1** |
| | Suppose that it is not true that $S_n = \arctan\left(\frac{n}{n+1}\right)$ for all values of $n$.<br>Then there is a smallest positive value, $k$ such that $S_k \neq \arctan\left(\frac{k}{k+1}\right)$. | |
| | Since $S_k > S_{k-1}$, $S_{k-1} = \arctan\left(\frac{k-1}{k}\right)$ and $\tan S_k = \frac{k}{k+1}$, but $S_k \neq \arctan\left(\frac{k}{k+1}\right)$<br>$S_k - S_{k-1} > \pi$. | **M1 M1** |
| | However, $S_k - S_{k-1} = \arctan\left(\frac{1}{2k^2}\right) < \frac{\pi}{2}$, so this is not possible. | **A1** |
| | Therefore $S_n = \arctan\left(\frac{n}{n+1}\right)$. | **A1** |
| | | |
| (ii) | $\tan 2\alpha_n = \frac{4n^2}{4n^4-1}$. | **M1 A1** |
| | So, $\frac{2\tan\alpha_n}{1-\tan^2\alpha_n} = \frac{4n^2}{4n^4-1}$ | **B1** |
| | Which simplifies to $2n^2\tan^2\alpha_n - (1-4n^2)\tan\alpha_n - 2n^2 = 0$ | **M1 A1** |
| | $(\tan\alpha_n + 2n^2)(2n^2\tan\alpha_n - 1) = 0$ | **A1** |
| | Since $\alpha_n$ must be acute, $\tan\alpha_n$ cannot equal $-2n^2$. | **B1** |
| | Therefore $\alpha_n = \arctan\left(\frac{1}{2n^2}\right)$. | |
| | $\sum_{n=1}^{\infty}\alpha_n = \lim_{n\to\infty} S_n = \arctan 1 = \frac{\pi}{4}$. | **M1 A1** |