STEP II, 2015, Q3 MS

Question 3

For the first part note that $T_8 - T_7$ can be interpreted as the triangles that can be made using the rod of length 8 and two other, shorter rods. These can then be counted by noting that there are 6 possibilities if the length 7 rod is used, 4 possibilities if the length 6 (but not the length 7) rod is used and 2 possibilities if the length 5 (but not 6 or 7) rod is used. It is clear that at least one rod longer than length 4 must be used. To evaluate $T_8 - T_6$ note that it is equal to $(T_8 - T_7) + (T_7 - T_6)$ and then evaluate $T_7 - T_6$ in a similar manner to $T_8 - T_7$. Similar reasoning easily gives formulae for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

For the induction, the rule for $T_{2m} - T_{2m-2}$ deduced in the previous part can be used to show the inductive step, while the easiest way to show the base case is to list the possibilities. The easiest way to establish the result for an odd number of rods is to use the formula for $T_{2m} - T_{2m-1}$ and the formula for T_{2m} that was just proven.

Question 3

	$T_8 - T_7$ is the number of triangles that can be made using a rod of length 8 and two other, shorter rods.	**M1**
	If the middle length rod has length 7 then the other rod can be 1, 2, 3, 4, 5 or 6.	**M1**
	If the middle length rod has length 6 then the other rod can be 2, 3, 4 or 5.	
	If the middle length rod has length 5 then the other rod can be 3 or 4.	**M1**
	$T_8 - T_7 = 2 + 4 + 6$.	**A1**
	Assume that the longest of the three rods has length 7:	**M1**
	If the middle length rod has length 6 then the other rod can be 1, 2, 3, 4 or 5.	**M1**
	If the middle length rod has length 5 then the other rod can be 2, 3 or 4.	
	If the middle length rod has length 4 then the other rod must be 3.	**M1**
	Therefore $T_7 - T_6 = 1 + 3 + 5$.	**A1**
	$T_8 - T_6 = T_8 - T_7 + T_7 - T_6 = 1 + 2 + 3 + 4 + 5 + 6$.	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1)$	**B1**
	$T_{2m} - T_{2m-2} = 1 + 2 + 3 + \cdots + 2(m-1)$	**B1**
	$T_4 = 3$. (The possibilities are $\{1, 2, 3\}, \{1, 3, 4\}$ and $\{2, 3, 4\}$.)	**B1**
	Substituting $m = 2$ into the equation gives $T_4 = \frac{1}{6}(2)(2-1)(4 \times 2 + 1) = 3$.	
	Therefore the formula is correct in the case $m = 2$.	**B1**
	Assume that the formula is correct in the case $m = k$: $T_{2(k+1)} = T_{2k} + \sum_{r=1}^{2k} r$	**M1**
	$T_{2(k+1)} = \frac{1}{6}k(k-1)(4k+1) + \frac{2k}{2}(2k+1)$	**M1**
	$T_{2(k+1)} = \frac{k}{6}[4k^2 - 3k - 1 + 12k + 6] = \frac{(k+1)}{6}(k)(4(k+1)+1)$, which is a statement of the formula where $m = k + 1$.	**M1**
	Therefore, by induction, $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1) = m(m-1)$.	**M1 A1**
	Therefore $T_{2m-1} = \frac{1}{6}m(m-1)(4m+1) - m(m-1)$.	
	$T_{2m-1} = \frac{1}{6}m(m-1)(4m-5)$. (Or $T_{2m+1} = \frac{1}{6}m(m+1)(4m-1)$)	**A1**