STEP II, 2015, Q1 MS

Question 1

For the first result, show that the gradient of the function is positive for all positive values of x (by differentiating) and also that $f(0) \geq 0$. Once this result has been established sum a set of the terms, using $x = \frac{1}{k}$, note that $\ln\left(1+\frac{1}{k}\right)$ can be written as $\ln(k+1) - \ln(k)$ and then the required result follows.

For the second part, first show that $x + \ln(1-x)$ is *negative* for $0 < x < 1$ and then use the substitution $x = \frac{1}{k^2}$, noting that $\ln\left(1-\frac{1}{k^2}\right)$ can be written as $\ln(k-1) - 2\ln(k) + \ln(k+1)$. Deal with the sum starting with $k = 2$ and then add the initial 1 afterwards.

Question 1

(i)	$\frac{d}{dx}(x - \ln(1+x)) = 1 - \frac{1}{1+x}$	**B1**
	For $x > 0$, $\frac{1}{1+x} < 1$	**M1**
	Therefore $\frac{d}{dx}(x - \ln(1+x)) > 0$ for $x > 0$	**A1**
	If $x = 0$, $x - \ln(1+x) = 0$	
	Therefore $x - \ln(1+x)$ is positive for all positive x.	**B1**
	Therefore $\frac{1}{k} - \ln\left(1+\frac{1}{k}\right) > 0$ for all positive k.	
	So, $\sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right)$	**B1**
	$\ln\left(1+\frac{1}{k}\right) = \ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k$	**M1**
	So, $\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln k = \ln(n+1) - \ln 1$	**M1**
	Therefore, $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$	**A1**

(ii)	$\frac{d}{dx}(x + \ln(1-x)) = 1 - \frac{1}{1-x}$	B1
	For $0 < x < 1$, $\frac{1}{1-x} > 1$	M1
	Therefore $\frac{d}{dx}(x + \ln(1-x)) < 0$ for $0 < x < 1$.	A1
	If $x = 0$, $x + \ln(1-x) = 0$	
	Therefore $x + \ln(1-x)$ is negative for $0 < x < 1$.	B1
	Therefore $\frac{1}{k^2} + \ln\left(1 - \frac{1}{k^2}\right) < 0$ for all $k > 1$.	
	So, $\sum_{k=2}^{n} \frac{1}{k^2} < \sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right)$	B1
	$-\ln\left(1 - \frac{1}{k^2}\right) = -\ln\left(\frac{k^2-1}{k^2}\right) = -\ln(k-1) + 2\ln k - \ln(k+1)$	M1 M1 A1
	So, $\sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right) = \ln 2 + \ln n - \ln(n+1)$	M1 A1
	As $n \to \infty$, $\ln n - \ln(n+1) \to 0$	B1
	Therefore, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \ln 2$	A1