STEP II, 2015, Q13 MS

Question 13

To calculate the expected value of the total cost, note that there is a constant component of ky and then the expected value of the $a(X-y)$ given that $X>y$ must be added, which can be calculated by integration of $(x-y)\lambda e^{-\lambda x}$ with respect to x, between y and ∞. Differentiating the expression for $E(C)$ with respect to y allows the position of the stationary point to be found. If this is at a negative value then y should be chosen to be 0 and otherwise the value of y for the stationary point should be used.

A slightly more complicated integration is needed to establish the formula for $Var(C)$ and then differentiation of this gives a value that is clearly negative for positive values of y, which shows that the variance is decreasing as y increases.

Question 13

(i)	$C = \begin{matrix} ky + a(X-y) & \text{for } X>y \\ ky & \text{for } X \le y \end{matrix}$	B1
	$E(C) = ky + a\int_y^\infty (x-y)\lambda e^{-\lambda x}dx$	M1 M1 A1
	Use the substitution $u = x - y$ in the integral:	
	$\int_y^\infty (x-y)\lambda e^{-\lambda x}dx = e^{-\lambda y}\int_0^\infty u\lambda e^{-\lambda u}du$	B1
	$\int_0^\infty u\lambda e^{-\lambda u}du = \left[-ue^{-\lambda u}\right]_0^\infty + \int_0^\infty e^{-\lambda u}du = \left[-ue^{-\lambda u} - \frac{1}{\lambda}e^{-\lambda u}\right]_0^\infty = \frac{1}{\lambda}$	M1
	Therefore $E(C) = ky + \frac{a}{\lambda}e^{-\lambda y}$.	A1
	$\frac{d}{dy}(E(C)) = k - ae^{-\lambda y}$, so the stationary point occurs when $y = \frac{1}{\lambda}\ln\frac{a}{k}$.	M1 A1
	If $\frac{a}{k} > 1$ then choose $y = \frac{1}{\lambda}\ln\frac{a}{k}$ as it is positive.	
	If $\frac{a}{k} \le 1$ then choose $y = 0$ as the minimum occurs at a negative value of y.	B1
(ii)	$E(C^2) = k^2y^2 + \int_y^\infty 2aky(x-y)\lambda e^{-\lambda x} + a^2(x-y)^2\lambda e^{-\lambda x}dx$	M1 A1
	Use the substitution $u = x - y$ in the integral:	
	$Integral = e^{-\lambda y}\int_0^\infty 2akyu\lambda e^{-\lambda u} + a^2u^2\lambda e^{-\lambda u}dx$	B1
	Applying integration done before: $\int_0^\infty 2akyu\lambda e^{-\lambda u}dx = \frac{2aky}{\lambda}$	
	Using integration by parts: $\int_0^\infty a^2u^2\lambda e^{-\lambda u}dx = \left[-a^2u^2e^{-\lambda u}\right]_0^\infty + \int_0^\infty \frac{2a^2u\lambda e^{-\lambda u}}{\lambda}dx$	M1 A1
	and, applying the integration already completed, $\int_0^\infty \frac{2a^2ue^{-\lambda u}}{\lambda}dx = \frac{2a^2}{\lambda^2}$.	
	Therefore $E(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y}$.	A1
	$\text{Var}(C^2) = E(C^2) - E(C)^2$	M1
	$\text{Var}(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y} - \left(ky + \frac{a}{\lambda}e^{-\lambda y}\right)^2$.	
	$\text{Var}(C^2) = \frac{a^2}{\lambda^2}\left(2e^{-\lambda y} - e^{-2\lambda y}\right)$.	A1
	$\frac{d}{dy}(\text{Var}(C^2)) = \frac{2a^2}{\lambda}e^{-\lambda y}\left(e^{-\lambda y} - 1\right)$	M1
	For $y > 0$, $\frac{d}{dy}(\text{Var}(C^2)) < 0$, so the variance decreases as y increases.	A1