STEP II, 2015, Q13

13 The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{for } x \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant.

It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geqslant 0$. If $X \leqslant y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$\mathrm{E}(C) = ky + \frac{a}{\lambda}\mathrm{e}^{-\lambda y}\,.$$

How should y be chosen in order to minimise $\mathrm{E}(C)$, in the different cases that arise according to the value of a/k?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.