STEP II, 2015, Q11 MS

Question 11

For the first part, the coordinates of A are found by applying simple trigonometric ratios and differentiation with respect to time gives the velocity of A. In the second part, the first equation results from consideration of conservation of momentum and the second results from conservation of energy (with a substitution based on the first equation made to eliminate one variable).

Since no energy is lost in any collisions the relationships from part (ii) must continue to hold and this shows that $\dot\theta$ cannot be 0 which means that the direction in which θ changes remains the same unless there is a collision. Since the first collision occurs when $\theta = 0$, the second one must be when $\theta = \pi$.

For the final part, note that the equations in part (ii) must still hold, and if $v = 0$, the kinetic energy of B must be 0. Since the kinetic energies of A and C must be equal (by symmetry) it must be the case that the kinetic energy of A is $\frac{1}{4}mu^2$ and can also be calculated from the expression for the velocity of A shown in part (i). Since $\dot\theta^2 > 0$, this can then be used to find the values of θ. Finally, note that given these values of θ, v will only be 0 on the occasions when $\dot\theta$ is positive.

Question 11

(i)	$A(x - a\cos\theta, a\sin\theta)$	**B1 B1**
	Differentiating: $(\dot x - a(-\sin\theta)\dot\theta, a(\cos\theta)\dot\theta)$	**M1**
	Since B is moving with velocity v and is at the point $(x, 0)$ at time t, $\dot x = v$:	
	Velocity of A is $(v + a\dot\theta\sin\theta, a\dot\theta\cos\theta)$.	**A1**
(ii)	Initial momentum was mu (horizontally).	**M1**
	Horizontal velocity of C will be the same as that of A, so horizontally the total momentum is given by $mv + 2m(v + a\dot\theta\sin\theta)$	**M1**
	Therefore $3v + 2a\dot\theta\sin\theta = u$.	**A1**
	Initial energy was $\frac{1}{2}mu^2$	**M1**
	Total energy is $\frac{1}{2}mv^2 + 2\left(\frac{1}{2}m\left(\left(v + a\dot\theta\sin\theta\right)^2 + \left(a\dot\theta\cos\theta\right)^2\right)\right)$	**M1 A1**
	Therefore $u^2 = v^2 + 2(v^2 + 2av\dot\theta\sin\theta + a^2\dot\theta^2\sin^2\theta + a^2\dot\theta^2\cos^2\theta)$	**M1**
	So $u^2 = 3v^2 + 4av\dot\theta\sin\theta + 2a^2\dot\theta^2$	
	Substituting $v = \frac{u - 2a\dot\theta\sin\theta}{3}$ gives $3u^2 = \left(u - 2a\dot\theta\sin\theta\right)^2 + 4a\dot\theta\sin\theta\left(u - 2a\dot\theta\sin\theta\right) + 6a^2\dot\theta^2$	**M1**
	$6a^2\dot\theta^2 = 3u^2 - u^2 + 4au\dot\theta\sin\theta - 4a^2\dot\theta^2\sin^2\theta - 4au\dot\theta\sin\theta + 8a^2\dot\theta^2\sin^2\theta$	
	$6a^2\dot\theta^2 - 4a^2\dot\theta^2\sin^2\theta = 2u^2$ So, $\dot\theta^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}$.	**A1**

(iii)	$\dot{\theta}^2 > 0$, so there can only be an instantaneous change of direction in which θ varies at a collision. Since the first collision will be when $\theta = 0$, the second collision must be when $\theta = \pi$.	B1 B1
(iv)	Since horizontal momentum must be mu, $v = 0 \implies 2a\dot{\theta}\sin\theta = u$.	B1
	The KE of A must be $\frac{1}{4}mu^2$, so $\frac{1}{2}ma^2\dot{\theta}^2 = \frac{1}{4}mu^2$	B1
	$\frac{1}{2}ma^2\dot{\theta}^2 = ma^2\dot{\theta}^2\sin^2\theta$	
	$\sin^2\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$	M1 A1
	v is only 0 when θ takes these values and $\dot{\theta}$ is positive as v would need a non-zero value to satisfy $3v + 2a\dot{\theta}\sin\theta = u$ if $\dot{\theta}$ is negative. (The relationship is still true since collisions are elastic).	B1