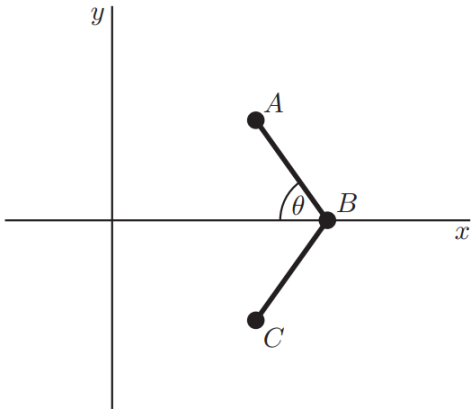
STEP II, 2015, Q11

11 Three particles, A, B and C, each of mass m, lie on a smooth horizontal table. Particles A and C are attached to the two ends of a light inextensible string of length $2a$ and particle B is attached to the midpoint of the string. Initially, A, B and C are at rest at points $(0, a)$, $(0, 0)$ and $(0, -a)$, respectively.

An impulse is delivered to B, imparting to it a speed u in the positive x direction. The string remains taut throughout the subsequent motion.



(i) At time t, the angle between the x-axis and the string joining A and B is θ, as shown in the diagram, and B is at $(x, 0)$. Write down the coordinates of A in terms of x, a and θ at this time. Given that the velocity of B is $(v, 0)$, show that the velocity of A is $(v + a\dot{\theta}\sin\theta\,,\ a\dot{\theta}\cos\theta)$, where the dot denotes differentiation with respect to time.

(ii) Show that, before A and C first collide,

$$3v + 2a\dot{\theta}\sin\theta = u \qquad \text{and} \qquad \dot{\theta}^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}\,.$$

(iii) When A and C collide, the collision is elastic (no energy is lost). At what value of θ does the second collision between particles A and C occur? (You should justify your answer.)

(iv) When $v = 0$, what are the possible values of θ? Is $v = 0$ whenever θ takes these values?