STEP II, 2014, Q8

8 For positive integers n, a and b, the integer c_r $(0 \leqslant r \leqslant n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a+bx)^n$. Write down an expression for c_r in terms of r, n, a and b.

For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \geqslant c_r$ for $0 \leqslant r \leqslant n$).

Show that

$$\frac{b(n+1)}{a+b} - 1 \leqslant m \leqslant \frac{b(n+1)}{a+b}\,.$$

Deduce that m is either a unique integer or one of two consecutive integers.

Let $\mathrm{G}(n,a,b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

(i) Evaluate $\mathrm{G}(9,1,3)$ and $\mathrm{G}(9,2,3)$.

(ii) For any positive integer k, find $\mathrm{G}(2k,a,a)$ and $\mathrm{G}(2k-1,a,a)$ in terms of k.

(iii) For fixed n and b, determine a value of a for which $\mathrm{G}(n,a,b)$ is greatest.

(iv) For fixed n, find the greatest possible value of $\mathrm{G}(n,1,b)$. For which values of b is this greatest value achieved?