STEP II, 2014, Q7 MS

Question 7:

By considering the regions $x \leq a$, $a < x < b$ and $x \geq b$, $f(x)$ can be written as

$$f(x) = \begin{matrix} a+b-2x & x \leq a \\ b-a & a<x<b \\ 2x-a-b & x \geq b \end{matrix}$$

Therefore the graph of $y = f(x)$ will be made up of two sloping sections (with gradients 2 and -2 and a horizontal section). The graph of $y = g(x)$ will have the same definition in the regions $x \leq a$ and $x \geq b$, with the sloping edges extending to a point of intersection on the x-axis. The quadrilateral with therefore have sides of equal length and right angles at each vertex, so it is a square.

In part (ii), sketches of the cases where $c = a$ and $c = b$ show that these cases give just one solution. If $a < c < b$ there will be no solutions and in the other regions there will be two solutions.

In part (iii) the graphs for the two sides of the equation can be related to graphs of the form of $g(x)$ (apart from the section which is replaced by a horizontal line) in the first part of the question. Since $d - c < b - a$, the horizontal sections of the two graphs must be at different heights so the number of solutions can be seen to be the same as the number of intersections of the graphs of the form of $g(x)$.