STEP II, 2014, Q6 MS

Question 6:

One of the standard trigonometric formulas can be used to show that

$$\sin\left(r+\tfrac{1}{2}\right)x - \sin\left(r-\tfrac{1}{2}\right)x = 2\cos rx \sin\tfrac{1}{2}x.$$

Summing these from $r = 1$ to $r = n$ will then give the required result.

In part (i), the definition can be rewritten as $S_2(x) = \sin x + \frac{1}{2}\sin 2x$. The stationary points can then be evaluated by differentiating the function. The sketch is then easy to complete.

For part (ii), differentiating the function gives $S_n'(x) = \cos x + \cos 2x + \cdots + \cos nx$. Applying the result from the start of the question, this can be written as

$$S_n'(x) = \frac{\sin\left(n+\frac{1}{2}x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

Since $\sin\frac{1}{2}x \neq 0$ in the given range, the stationary points are where $\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x = 0$. This can then be simplified to the required form by splitting $\sin\left(n+\frac{1}{2}\right)x$ into functions of nx and $\frac{1}{2}x$ and noting that $\sin\frac{1}{2}x \neq 0$ and $\cos\frac{1}{2}x \neq 0$ in the given range, so both can be divided by. By noting that the difference between $S_{n-1}(x)$ and $S_n(x)$ is $\frac{1}{n}\sin nx$ the result just shown can be used to show the final result of part (ii). Part (iii) then follows by induction.