STEP II, 2014, Q3 MS

Question 3:

By drawing a diagram and marking the shortest distance a pair of similar triangles can be used to show that $\frac{c/m}{c\sqrt{m^2+1}/m} = \frac{d}{c}$, which simplifies to $d = c/\sqrt{m^2+1}$.

For the second part, the tangent to the curve at the general point (x, y) will have a gradient of y' and so the y-intercept will be at the point $(0, y - xy')$. Therefore the result from part (i) can be applied using $m = y'$ and $c = y - xy'$ to give $a = (y - xy')/\sqrt{(y')^2 + 1}$, which rearranges to give the required result.

Differentiating the equation then gives $y''\left(a^2y' + x(y - xy')\right) = 0$ and so either $y'' = 0$ or $a^2y' + x(y - xy') = 0$.

If $y'' = 0$ then the equation will be of a straight line and the y-intercept can be deduced in terms of m.

If $a^2y' + x(y - xy') = 0$, then the differential equation can be solved to give the equation of a circle.

Part (iii) then requires combining the two possible cases from part (ii) to construct a curve which satisfies the conditions given. This must be an arc of a circle with no vertical tangents, with straight lines at either end of the arc in the direction of the tangents to the circle at that point.