STEP II, 2014, Q3

3 **(i)** Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geqslant 0$, is $c(m^2+1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C.

Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2\left(1 + (y')^2\right). \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1+m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.