STEP II, 2014, Q2 MS

Question 2:

By rewriting in terms of $\cos 2nx$ it can be shown that $\int_0^\pi \sin^2 nx\, dx = \frac{\pi}{2}$ and $\int_0^\pi n^2 \cos^2 nx\, dx = \frac{n^2\pi}{2}$. Therefore (*) must be satisfied as n is a positive integer. The function $f(x) = x$ does not satisfy (*) and $f(0) = 0$ but $f(\pi) \neq 0$. The function $g(x) = f(\pi - x)$ will therefore provide a counterexample where $g(\pi) = 0$, but $g(0) \neq 0$.

In part (ii), $f(x) = x^2 - \pi x$ will need to be selected to be able to use the assumption that (*) is satisfied. The two sides of (*) can then be evaluated:

$$\int_0^\pi x^4 - 2\pi x^3 + \pi^2 x^2\, dx = \frac{\pi^5}{30}$$

$$\int_0^\pi 4x^2 - 4\pi x + \pi^2\, dx = \frac{\pi^3}{3}$$

Substitution into (*) then leads to the inequality $\pi^2 \leq 10$.

To satisfy the conditions on $f(x)$ for the second type of function, the values of p, q and r must satisfy $q + r = 0$ and $p + r = 0$. Evaluating the integrals then leads to $\pi \leq \frac{22}{7}$.

Since $\left(\frac{22}{7}\right)^2 < 10$, $\pi \leq \frac{22}{7}$ leads to a better estimate for π^2.