STEP II, 2014, Q2

2 This question concerns the inequality

$$\int_0^{\pi} \left(\mathrm{f}(x)\right)^2 \mathrm{d}x \leqslant \int_0^{\pi} \left(\mathrm{f}'(x)\right)^2 \mathrm{d}x\,. \qquad (*)$$

(i) Show that $(*)$ is satisfied in the case $\mathrm{f}(x) = \sin nx$, where n is a positive integer.

Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $\mathrm{f}(0) \neq 0$ or $\mathrm{f}(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $\mathrm{f}(0) = \mathrm{f}(\pi) = 0$.

By setting $\mathrm{f}(x) = ax^2 + bx + c$, where a, b and c are suitably chosen, show that $\pi^2 \leqslant 10$.

By setting $\mathrm{f}(x) = p\sin\frac{1}{2}x + q\cos\frac{1}{2}x + r$, where p, q and r are suitably chosen, obtain another inequality for π.

Which of these inequalities leads to a better estimate for π^2?