STEP II, 2014, Q1 MS

Question 1:

Drawing a diagram and considering the horizontal and vertical distances will establish the relationships for $x\cos\theta$ and $x\sin\theta$ easily. The quadratic equation will then follow from use of the identity $\cos^2\theta + \sin^2\theta \equiv 1$. The same reasoning applied to a diagram showing the case where P and Q lie on AC produced and BC produced will show that the same equation is satisfied.

(*) will be linear if the coefficient of x^2 is 0, so therefore $\cos(\alpha+\beta)$ will need to equal $-\frac{1}{2}$, which gives a relationship between α and β. For (*) to have distinct roots the discriminant must be positive. Using some trigonometric identities it can be shown that the discriminant is equal to $4(1-(\sin\alpha - \sin\beta)^2)$ and it should be easy to explain why this must be greater than 0.

The first case in part (iii) leads to $x = \sqrt{2} \pm 1$ and so there are two diagrams to be drawn. In each case the line joining P to Q will be horizontal.

The second case in part (iii) is an example where (*) is linear. This leads to $x = \frac{\sqrt{3}}{3}$. Therefore Q is at the same point as C and so the point P is the midpoint of AC.