STEP II, 2014, Q13

13 A random number generator prints out a sequence of integers I_1, I_2, I_3, ... Each integer is independently equally likely to be any one of 1, 2, ..., n, where n is fixed. The random variable X takes the value r, where I_r is the first integer which is a repeat of some earlier integer.

Write down an expression for $\mathrm{P}(X = 4)$.

(i) Find an expression for $\mathrm{P}(X = r)$, where $2 \leqslant r \leqslant n+1$. Hence show that, for any positive integer n,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots = 1 .$$

(ii) Write down an expression for $\mathrm{E}(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $\mathrm{P}(X \geqslant k)$.

(iv) Show that, for any discrete random variable Y taking the values 1, 2, ..., N,

$$\mathrm{E}(Y) = \sum_{k=1}^{N} \mathrm{P}(Y \geqslant k) .$$

Hence show that, for any positive integer n,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right) + \cdots = 0.$$