STEP II, 2014, Q12

12 The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $\mathrm{f}(t)$ and cumulative distribution function $\mathrm{F}(t)$. The *hazard function*, $\mathrm{h}(t)$, is defined, for $\mathrm{F}(t) < 1$, by

$$\mathrm{h}(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)}\,.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $\mathrm{h}(t)\,\delta t$ for small values of δt.

(ii) Find the hazard function in the case $\mathrm{F}(t) = t/a$ for $0 < t < a$. Sketch $\mathrm{f}(t)$ and $\mathrm{h}(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $\mathrm{h}(t)$ is constant for $t > b$ and zero otherwise if and only if $\mathrm{f}(t) = k\mathrm{e}^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$\mathrm{h}(t) = \left(\frac{\lambda}{\theta^{\lambda}}\right) t^{\lambda - 1}\,,$$

where λ and θ are positive constants. Find the probability density function for T.