STEP II, 2014, Q1

1 In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leqslant \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$\big(1 + 2\cos(\alpha+\beta)\big)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0\,. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leqslant \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45°$ and (b) $\alpha = 30°$, $\beta = 90°$, and illustrate each case with a sketch.