

## STEP II, 2013, Q8

8 The function  $f$  satisfies  $f(x) > 0$  for  $x \geq 0$  and is strictly decreasing (which means that  $f(b) < f(a)$  for  $b > a$ ).

(i) For  $t \geq 0$ , let  $A_0(t)$  be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve  $y = f(x)$ , the  $y$ -axis and the line  $y = f(t)$ . Show that  $A_0(t)$  can be written in the form

$$A_0(t) = x_0 (f(x_0) - f(t)),$$

where  $x_0$  satisfies  $x_0 f'(x_0) + f(x_0) = f(t)$ .

(ii) The function  $g$  is defined, for  $t > 0$ , by

$$g(t) = \frac{1}{t} \int_0^t f(x) dx.$$

Show that  $tg'(t) = f(t) - g(t)$ .

Making use of a sketch show that, for  $t > 0$ ,

$$\int_0^t (f(x) - f(t)) dx > A_0(t)$$

and deduce that  $-t^2 g'(t) > A_0(t)$ .

(iii) In the case  $f(x) = \frac{1}{1+x}$ , use the above to establish the inequality

$$\ln \sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}},$$

for  $t > 0$ .



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