STEP II, 2013, Q8

8 The function f satisfies $f(x) > 0$ for $x \geqslant 0$ and is strictly decreasing (which means that $f(b) < f(a)$ for $b > a$).

(i) For $t \geqslant 0$, let $A_0(t)$ be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve $y = f(x)$, the y-axis and the line $y = f(t)$. Show that $A_0(t)$ can be written in the form

$$A_0(t) = x_0\left(f(x_0) - f(t)\right),$$

where x_0 satisfies $x_0 f'(x_0) + f(x_0) = f(t)$.

(ii) The function g is defined, for $t > 0$, by

$$g(t) = \frac{1}{t}\int_0^t f(x)\,dx.$$

Show that $tg'(t) = f(t) - g(t)$.

Making use of a sketch show that, for $t > 0$,

$$\int_0^t \left(f(x) - f(t)\right) dx > A_0(t)$$

and deduce that $-t^2 g'(t) > A_0(t)$.

(iii) In the case $f(x) = \dfrac{1}{1+x}$, use the above to establish the inequality

$$\ln\sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}},$$

for $t > 0$.