STEP II, 2013, Q6 MS

Question 6.

The definition of the sequence can be used to find a relationship between u_{n+2} and u_n and therefore also a relationship between u_n and u_{n-2}. Taking the difference of these then leads to the required result.

It is clear from the definition of the sequence that, if one term is between 1 and 2, then the next term will also be between 1 and 2. This is then easy to present in the form of a proof by induction for part (ii).

The result of part (i) shows that the sequence in part (iii) is increasing and the result proved in part (ii) shows that it is bounded above. The theorem provided at the start of the question therefore shows that the sequence converges. Similarly the second sequence is bounded below and decreasing (and therefore if the terms are all multiplied by -1 a sequence will be generated which is bounded above and increasing). Therefore the second sequence also converges to a limit.

The relationship between u_n and u_{n-2} established in part (i) can then be used to find the value of this limit and, as it is the same for both the odd terms and the even terms, the sequence must tend to the same limit as well.

Finally, starting the sequence at 3 will still lead to the same conclusion as the next term will be between 1 and 2 and all further terms will also be within that range, so all of the arguments will still hold for this new sequence.