

## STEP II, 2013, Q6

- 6 In this question, the following theorem may be used.  
*Let  $u_1, u_2, \dots$  be a sequence of (real) numbers. If the sequence is bounded above (that is,  $u_n \leq b$  for all  $n$ , where  $b$  is some fixed number) and increasing (that is,  $u_n \geq u_{n-1}$  for all  $n$ ), then the sequence tends to a limit (that is, converges).*

The sequence  $u_1, u_2, \dots$  is defined by  $u_1 = 1$  and

$$u_{n+1} = 1 + \frac{1}{u_n} \quad (n \geq 1). \quad (*)$$

- (i) Show that, for  $n \geq 3$ ,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}.$$

- (ii) Prove, by induction or otherwise, that  $1 \leq u_n \leq 2$  for all  $n$ .
- (iii) Show that the sequence  $u_1, u_3, u_5, \dots$  tends to a limit, and that the sequence  $u_2, u_4, u_6, \dots$  tends to a limit. Find these limits and deduce that the sequence  $u_1, u_2, u_3, \dots$  tends to a limit.

Would this conclusion change if the sequence were defined by (\*) and  $u_1 = 3$ ?



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