STEP II, 2013, Q6

6 In this question, the following theorem may be used.

Let u_1, u_2, ... be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leqslant b$ for all n, where b is some fixed number) and increasing (that is, $u_n \geqslant u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).

The sequence u_1, u_2, ... is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \qquad (n \geqslant 1). \qquad (*)$$

(i) Show that, for $n \geqslant 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leqslant u_n \leqslant 2$ for all n.

(iii) Show that the sequence u_1, u_3, u_5, ... tends to a limit, and that the sequence u_2, u_4, u_6, ... tends to a limit. Find these limits and deduce that the sequence u_1, u_2, u_3, ... tends to a limit.

Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?