

STEP II, 2013, Q2

2 For $n \geq 0$, let

$$I_n = \int_0^1 x^n(1-x)^n dx.$$

(i) For $n \geq 1$, show by means of a substitution that

$$\int_0^1 x^{n-1}(1-x)^n dx = \int_0^1 x^n(1-x)^{n-1} dx$$

and deduce that

$$2 \int_0^1 x^{n-1}(1-x)^n dx = I_{n-1}.$$

Show also, for $n \geq 1$, that

$$I_n = \frac{n}{n+1} \int_0^1 x^{n-1}(1-x)^{n+1} dx$$

and hence that $I_n = \frac{n}{2(2n+1)} I_{n-1}$.

(ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!}.$$

(iii) Use the substitution $x = \sin^2 \theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.



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