STEP II, 2013, Q2

2 For $n \geqslant 0$, let

$$I_n = \int_0^1 x^n(1-x)^n \mathrm{d}x\,.$$

(i) For $n \geqslant 1$, show by means of a substitution that

$$\int_0^1 x^{n-1}(1-x)^n \mathrm{d}x = \int_0^1 x^n(1-x)^{n-1}\mathrm{d}x$$

and deduce that

$$2\int_0^1 x^{n-1}(1-x)^n \mathrm{d}x = I_{n-1}\,.$$

Show also, for $n \geqslant 1$, that

$$I_n = \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1}\mathrm{d}x$$

and hence that $I_n = \dfrac{n}{2(2n+1)} I_{n-1}\,.$

(ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!}\,.$$

(iii) Use the substitution $x = \sin^2\theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.