STEP II, 2013, Q13

13 A biased coin has probability p of showing a head and probability q of showing a tail, where $p \neq 0$, $q \neq 0$ and $p \neq q$. When the coin is tossed repeatedly, runs occur. A *straight run* of length n is a sequence of n consecutive heads or n consecutive tails. An *alternating run* of length n is a sequence of length n alternating between heads and tails. An alternating run can start with either a head or a tail.

Let S be the length of the longest straight run beginning with the first toss and let A be the length of the longest alternating run beginning with the first toss.

(i) Explain why $\mathrm{P}(A = 1) = p^2 + q^2$ and find $\mathrm{P}(S = 1)$. Show that $\mathrm{P}(S = 1) < \mathrm{P}(A = 1)$.

(ii) Show that $\mathrm{P}(S = 2) = \mathrm{P}(A = 2)$ and determine the relationship between $\mathrm{P}(S = 3)$ and $\mathrm{P}(A = 3)$.

(iii) Show that, for $n > 1$, $\mathrm{P}(S = 2n) > \mathrm{P}(A = 2n)$ and determine the corresponding relationship between $\mathrm{P}(S = 2n + 1)$ and $\mathrm{P}(A = 2n + 1)$. [You are advised *not* to use $p + q = 1$ in this part.]