STEP II, 2012, Q9 MS

Question 9

In the standard way, we use the constant-acceleration formulae to get

$$x = ut\cos\alpha \text{ and } y = 2h - ut\sin\alpha - \tfrac{1}{2}gt^2 .$$

When $x = a$, $t = \dfrac{a}{u\cos\alpha}$. Substituting this into the equation for $y \Rightarrow y = 2h - a\tan\alpha - \dfrac{ga^2}{2u^2}\sec^2\alpha$.

As $y > h$ at this point (the ball, assuming it to be "a particle", is above the net), we get

$h - a\tan\alpha > \dfrac{ga^2}{2u^2}\sec^2\alpha \Rightarrow \dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}$, as required.

For the next part, we set $y = 0$ in $y = 2h - ut\sin\alpha - \frac{1}{2}gt^2$ and solve as a quadratic in t to get

$t = \dfrac{-2u\sin\alpha + \sqrt{4u^2\sin^2\alpha + 16gh}}{2g}$ ... (the positive root is required).

Setting $x = (u\cos\alpha)t$ and noting that $x < b$, $u\cos\alpha\left(\dfrac{\sqrt{u^2\sin^2\alpha + 4gh} - u\sin\alpha}{g}\right) < b$

$\Rightarrow \sqrt{u^2\sin^2\alpha + 4gh} < \dfrac{bg}{u\cos\alpha} + u\sin\alpha$.

There are several ways to proceed from here, but this is (perhaps) the most straightforward.

Squaring $\Rightarrow u^2\sin^2\alpha + 4gh < \dfrac{b^2g^2\sec^2\alpha}{u^2} + 2bg\tan\alpha + u^2\sin^2\alpha$

Cancelling $u^2\sin^2\alpha$ both sides & dividing by $g \Rightarrow 4h < \dfrac{b^2g\sec^2\alpha}{u^2} + 2b\tan\alpha$

Re-arranging for $\dfrac{1}{u^2} \Rightarrow \dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{1}{u^2}$

Using the first result, $\dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$, in here $\Rightarrow \dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$

Re-arranging for $\tan\alpha \Rightarrow ab(b-a)\tan\alpha < h(b^2 - 2a^2)$, which leads to the required final answer $\tan\alpha < \dfrac{h(b^2 - 2a^2)}{ab(b-a)}$. However, it is necessary (since we might otherwise be dividing by a quantity that could be negative) to explain that $b > a$ (we are now on the other side of the net to the projection point) else the direction of the inequality would reverse.