STEP II, 2012, Q8 MS

Question 8

$\beta - \alpha > q\ (>0) \Rightarrow \beta^2 - 2\alpha\beta + \alpha^2 > q^2 \Rightarrow \alpha^2 + \beta^2 - q^2 > 2\alpha\beta \Rightarrow \dfrac{\alpha^2+\beta^2-q^2}{\alpha\beta} > 2 \Rightarrow$ the opening result, $\dfrac{\alpha^2+\beta^2-q^2}{\alpha\beta} - 2 > 0$.

$u_{n+1} = \dfrac{u_n{}^2 - q^2}{u_{n-1}}$ etc. $\Rightarrow u_n{}^2 - u_{n+1}u_{n-1} = q^2 = u_{n+1}{}^2 - u_{n+2}u_n$ (since the result is true at all stages) and equating for $q^2 \Rightarrow u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$.

Now this gives $\dfrac{u_n + u_{n+2}}{u_{n+1}} = \dfrac{u_{n-1}+u_{n+1}}{u_n}$ which $\Rightarrow \dfrac{u_{n-1}+u_{n+1}}{u_n}$ is constant (independent of n). Calling this constant p gives $u_{n+1} - pu_n + u_{n-1} = 0$, as required. In order to determine p, we only need to use the fact that $p = \dfrac{u_{n-1}+u_{n+1}}{u_n}$ for all n, so we choose the first few terms to work with.

$$u_2 = \frac{\beta^2 - q^2}{\alpha} \text{ and } p = \frac{u_0 + u_2}{u_1} = \frac{\alpha + \dfrac{\beta^2-q^2}{\alpha}}{\beta} = \frac{\alpha^2+\beta^2-q^2}{\alpha\beta}.$$

Alternatively, $u_2 = \gamma = \dfrac{\beta^2-q^2}{\alpha} = p\beta - \alpha \Leftrightarrow p = \dfrac{\alpha^2+\beta^2-q^2}{\alpha\beta}$

and $u_3 = \dfrac{\gamma^2 - q^2}{\beta} = p\gamma - \beta \Leftrightarrow p = \dfrac{\gamma^2+\beta^2-q^2}{\beta\gamma} = \dfrac{\left(\dfrac{\beta^2-q^2}{\alpha}\right)^2 + \beta^2 - q^2}{\beta\left(\dfrac{\beta^2-q^2}{\alpha}\right)}$

$$= \frac{(\beta^2-q^2)^2 + \alpha^2(\beta^2-q^2)}{\alpha\beta(\beta^2-q^2)} = \frac{\beta^2-q^2+\alpha^2}{\alpha\beta}$$

since $\beta^2 - q^2 \neq 0$ as u_2 non-zero (given). Since p is consistent for any chosen α, β, the proof follows inductively on any two consecutive terms of the sequence.

Finally, on to the given cases.

If $\beta > \alpha + q$, $u_{n+1} - u_n = (p-1)u_n - u_{n-1} = \left(\dfrac{\beta^2+\alpha^2-q^2}{\alpha\beta} - 1\right)u_n - u_{n-1}$

$> (2-1)u_n - u_{n-1}$ by the initial result

$> u_n - u_{n-1}$

Hence, if $u_n - u_{n-1} > 0$ then so is $u_{n+1} - u_n$. Since $\beta > \alpha$, $u_2 - u_1 > 0$ and proof follows inductively.

If $\beta = \alpha + q$ then $p = 2$ and $u_{n+1} - u_n = u_n - u_{n-1}$ so that the sequence is an AP.
Also, $u_0 = \alpha$, $u_1 = \alpha + q$, $u_2 = \alpha + 2q$, … $\Rightarrow$ the common difference is q (and we still have a strictly increasing sequence, since $q > 0$ given).