STEP II, 2012 Q8

8 The positive numbers α, β and q satisfy $\beta - \alpha > q$. Show that

$$\frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0\,.$$

The sequence u_0, u_1, ... is defined by $u_0 = \alpha$, $u_1 = \beta$ and

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \qquad (n \geqslant 1),$$

where α, β and q are given positive numbers (and α and β are such that no term in the sequence is zero). Prove that $u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$. Prove also that

$$u_{n+1} - pu_n + u_{n-1} = 0$$

for some number p which you should express in terms of α, β and q.

Hence, or otherwise, show that if $\beta > \alpha + q$, the sequence is strictly increasing (that is, $u_{n+1} - u_n > 0$ for all n). Comment on the case $\beta = \alpha + q$.