STEP II, 2012, Q7 MS

Question 7

Many of you will know that this point G, used here, is the centroid of the triangle, and has position vector $\mathbf{g} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

Then $\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \frac{1}{3}(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$ and so $\overrightarrow{GY_1} = -\frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$, where $\lambda_1 > 0$.

Also $\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) = \frac{1}{3}([1-2\lambda]\mathbf{x}_1 + [1+\lambda_1](\mathbf{x}_2 + \mathbf{x}_3))$, the first printed result.

The really critical observation here is that the circle centre O, radius 1 has equation $|\,\mathbf{x}\,|^2 = 1$ or $\mathbf{x}\,.\,\mathbf{x} = 1$, where $\mathbf{x}$ can be the p.v. of any point on the circle.

Thus, since $\overrightarrow{OY_1} \centerdot \overrightarrow{OY_1} = 1$, we have

$$1 = \tfrac{1}{9}\left\{(1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \centerdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2 \mathbf{x}_2 \centerdot \mathbf{x}_3\right\}$$

$$\Rightarrow \quad 9 = 1 - 4\lambda_1 + 4\lambda_1^2 + 2 + 4\lambda_1 + 2\lambda_1^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \centerdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2 \mathbf{x}_2 \centerdot \mathbf{x}_3$$

$$\Rightarrow \quad 0 = -3(1-\lambda_1)(1+\lambda_1) + (1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \centerdot (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)^2 \mathbf{x}_2 \centerdot \mathbf{x}_3$$

As $\lambda_1 > 0$, $0 = -3(1-\lambda_1) + (1-2\lambda_1)\mathbf{x}_1 \centerdot (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)\mathbf{x}_2 \centerdot \mathbf{x}_3$

$$\Rightarrow \quad 0 = -3 + 3\lambda_1 + (\mathbf{x}_1 \centerdot \mathbf{x}_2 + \mathbf{x}_2 \centerdot \mathbf{x}_3 + \mathbf{x}_3 \centerdot \mathbf{x}_1) + \lambda_1(\mathbf{x}_2 \centerdot \mathbf{x}_3) - 2\lambda_1(\mathbf{x}_1 \centerdot \mathbf{x}_2 + \mathbf{x}_1 \centerdot \mathbf{x}_3)$$

$\Rightarrow \quad \lambda_1 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \alpha - 2\beta - 2\gamma}$, using $\alpha = \mathbf{x}_2 \bullet \mathbf{x}_3$, $\beta = \mathbf{x}_3 \bullet \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \bullet \mathbf{x}_2$.

Similarly, $\lambda_2 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \beta - 2\alpha - 2\gamma}$ and $\lambda_3 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \gamma - 2\alpha - 2\beta}$.

Using $\dfrac{GX_i}{GY_i} = \dfrac{1}{\lambda_i}$ $(i = 1, 2, 3)$, $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_3} = \dfrac{9 + (\alpha + \beta + \gamma) - 4(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)}$

$$= \frac{9 - 3(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)} = 3.$$

[Interestingly, this result generalises to n points on a circle: $\displaystyle\sum_{i=1}^{n} \frac{GX_i}{GY_i} = n$.]