

STEP II, 2012, Q7 MS

Question 7

Many of you will know that this point G , used here, is the centroid of the triangle, and has position vector $\mathbf{g} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

Then $\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \frac{1}{3}(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$ and so $\overrightarrow{GY_1} = -\frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$, where $\lambda_1 > 0$.

Also $\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) = \frac{1}{3}([1 - 2\lambda_1]\mathbf{x}_1 + [1 + \lambda_1](\mathbf{x}_2 + \mathbf{x}_3))$, the first printed result.

The really critical observation here is that the circle centre O , radius 1 has equation $|\mathbf{x}|^2 = 1$ or $\mathbf{x} \cdot \mathbf{x} = 1$, where \mathbf{x} can be the p.v. of any point on the circle.

Thus, since $\overrightarrow{OY_1} \cdot \overrightarrow{OY_1} = 1$, we have

$$\begin{aligned} 1 &= \frac{1}{9} \{ (1 - 2\lambda_1)^2 + 2(1 + \lambda_1)^2 + 2(1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \} \\ \Rightarrow 9 &= 1 - 4\lambda_1 + 4\lambda_1^2 + 2 + 4\lambda_1 + 2\lambda_1^2 + 2(1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \\ \Rightarrow 0 &= -3(1 - \lambda_1)(1 + \lambda_1) + (1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + (1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \\ \text{As } \lambda_1 > 0, \quad 0 &= -3(1 - \lambda_1) + (1 - 2\lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + (1 + \lambda_1)\mathbf{x}_2 \cdot \mathbf{x}_3 \\ \Rightarrow 0 &= -3 + 3\lambda_1 + (\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_2 \cdot \mathbf{x}_3 + \mathbf{x}_3 \cdot \mathbf{x}_1) + \lambda_1(\mathbf{x}_2 \cdot \mathbf{x}_3) - 2\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \cdot \mathbf{x}_3) \\ \Rightarrow \lambda_1 &= \frac{3 - (\alpha + \beta + \gamma)}{3 + \alpha - 2\beta - 2\gamma}, \text{ using } \alpha = \mathbf{x}_2 \cdot \mathbf{x}_3, \beta = \mathbf{x}_3 \cdot \mathbf{x}_1 \text{ and } \gamma = \mathbf{x}_1 \cdot \mathbf{x}_2. \end{aligned}$$

$$\text{Similarly, } \lambda_2 = \frac{3 - (\alpha + \beta + \gamma)}{3 + \beta - 2\alpha - 2\gamma} \text{ and } \lambda_3 = \frac{3 - (\alpha + \beta + \gamma)}{3 + \gamma - 2\alpha - 2\beta}.$$

$$\begin{aligned} \text{Using } \frac{GX_i}{GY_i} &= \frac{1}{\lambda_i} \quad (i = 1, 2, 3), \quad \frac{GX_1}{GY_1} + \frac{GX_2}{GY_2} + \frac{GX_3}{GY_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{9 + (\alpha + \beta + \gamma) - 4(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)} \\ &= \frac{9 - 3(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)} = 3. \end{aligned}$$

[Interestingly, this result generalises to n points on a circle: $\sum_{i=1}^n \frac{GX_i}{GY_i} = n$.]



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