STEP II, 2012 Q7

7 Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly.

Given that $\overrightarrow{GY_1} = -\lambda_1 \overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1} = \tfrac{1}{3}\big((1 - 2\lambda_1)\mathbf{x}_1 + (1 + \lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\big)$$

and hence that

$$\lambda_1 = \frac{3 - \alpha - \beta - \gamma}{3 + \alpha - 2\beta - 2\gamma},$$

where $\alpha = \mathbf{x}_2 \,.\, \mathbf{x}_3$, $\beta = \mathbf{x}_3 \,.\, \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \,.\, \mathbf{x}_2$.

Deduce that $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = 3$.