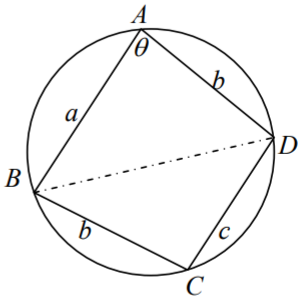
STEP II, 2012, Q6 MS

Question 6



A quick diagram helps here, leading to the important observation, from the GCSE geometry result "*opposite angles of a cyclic quad. add to 180°*", that $\angle BCD = 180^\circ - \theta$. Then, using the Cosine Rule twice (and noting that $\cos(180^\circ - \theta) = -\cos\theta$):

in ΔBAD: $BD^2 = a^2 + d^2 - 2ad\cos\theta$

in ΔBCD: $BD^2 = b^2 + c^2 + 2bc\cos\theta$

Equating for BD^2 and re-arranging gives $\cos\theta = \dfrac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}$

Next, the well-known formula for triangle area, $\Delta = \frac{1}{2}ab\sin C$, twice, gives $Q = \frac{1}{2}ad\sin\theta + \frac{1}{2}bc\sin\theta$, since $\sin(\pi - \theta) = \sin\theta$. Rearranging then gives $\sin\theta = \dfrac{2Q}{ad + bc}$ or $\dfrac{4Q}{2(ad + bc)}$.

Use of $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \dfrac{16Q^2}{4(ad + bc)^2} + \dfrac{(a^2 - b^2 - c^2 + d^2)^2}{4(ad + bc)^2} = 1$ and this then gives the printed result, $16Q^2 = 4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.

Then, $16Q^2 = (2ad + 2bc - a^2 + b^2 + c^2 - d^2)(2ad + 2bc + a^2 - b^2 - c^2 + d^2)$ by the *difference-of-two-squares* factorisation

$$= ([b + c]^2 - [a - d]^2)([a + d]^2 - [b - c]^2)$$
$$= ([b + c] - [a - d])([b + c] + [a - d])([a + d] - [b - c])([a + d] + [b - c])$$

using the *difference-of-two-squares* factorisation in each large bracket

$$= (b + c + d - a)(a + b + c - d)(a + c + d - b)(a + b + d - c).$$

Splitting the 16 into four 2's (one per bracket) and using $2s = a + b + c + d$

$$\Rightarrow Q^2 = \frac{(2s - 2a)}{2}\frac{(2s - 2b)}{2}\frac{(2s - 2c)}{2}\frac{(2s - 2d)}{2} = (s - a)(s - b)(s - c)(s - d).$$

Finally, for a triangle (guaranteed cyclic), letting $d \to 0$ (**Or** $s - d \to s$ **Or** let $D \to A$), we get the result known as *Heron's Formula*: $\Delta = \sqrt{s(s - a)(s - b)(s - c)}$.