STEP II, 2012, Q5 MS

Question 5

With any curve-sketching question of this kind, it is important to grasp those features that are important and ignore those that aren't. For instance, throughout this question, the position of the y-axis is entirely immaterial: it could be drawn through any branch of the curves in question or, indeed, appear as an asymptote. So the usually key detail of the y-intercept, at $\left(0, \frac{1}{a^2-1}\right)$ in part (i), does not help decide what the function is up to. The asymptotes, turning points (clearly important in part (ii) since they are specifically requested), and any symmetries are important. The other key features to decide upon are the "short-term" (when x is small) and the "long-term" (as $x \to \pm\infty$) behaviours.

In (i), there are vertical asymptotes at $x = a - 1$ and $x = a + 1$; while the x-axis is a horizontal asymptote. There is symmetry in the line $x = a$ (a consequence of which is the maximum TP in the "middle" branch) and the "long-term" behaviour of the curve is that it ultimately resembles the graph of $y = \frac{1}{x^2}$.

(ii) Differentiating the function in (ii) gives

$$g'(x) = \frac{-2}{\left[(x-a)^2-1\right]^2\left[(x-b)^2-1\right]^2}\left\{(x-b)\left[(x-a)^2-1\right]+(x-a)\left[(x-b)^2-1\right]\right\}$$

and setting the numerator $= 0 \Rightarrow (x-a)(x-b)\left[x-a+x-b\right]+\left[x-a+x-b\right]=0$. Factorising yields $(2x-a-b)\left(x^2-(a+b)x+(ab-1)\right)=0$, so that $x = \frac{1}{2}(a+b)$ or $\frac{a+b\pm\sqrt{(a+b)^2-4ab+4}}{2}$.

In the first case, where $b > a + 2$ (i.e. $a + 1 < b - 1$), there are five branches of the curve, with 4 vertical asymptotes: $x = a \pm 1$ and $x = b \pm 1$. As the function changes sign as it "crosses" each asymptote, and the "long-term" behaviour is still to resemble $y = \frac{1}{x^2}$, these branches alternate above and below the x-axis, with symmetry in $x = \frac{1}{2}(a+b)$.

In the second case, where $b = a + 2$ (i.e. $a + 1 = b - 1$), the very middle section has collapsed, leaving only the four branches, but the curve is otherwise essentially unchanged from the previous case.