STEP II, 2012, Q4 MS

Question 4

(i) This first result is easily established: For $n, k > 1$, $n^{k+1} > n^k$ and $k+1 > k$ so $(k+1) \times n^{k+1} > k \times n^k$

$\Rightarrow \dfrac{1}{(k+1)n^{k+1}} < \dfrac{1}{kn^k}$ (since all terms are positive).

Then $\ln\left(1+\dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \dfrac{1}{4n^4} + \dfrac{1}{5n^5} - \ldots$ (a result which is valid since $0 < \dfrac{1}{n} < 1$)

$= \dfrac{1}{n} - \left(\dfrac{1}{2n^2} - \dfrac{1}{3n^3}\right) - \left(\dfrac{1}{4n^4} - \dfrac{1}{5n^5}\right) - \ldots < \dfrac{1}{n}$ since each bracketed term is positive, using

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the previous result. Exponentiating then gives $1 + \dfrac{1}{n} < e^{\frac{1}{n}} \Rightarrow \left(1+\dfrac{1}{n}\right)^n < e$.

(ii) A bit of preliminary log. work enables us to use the $\ln(1+x)$ result on

$$\ln\left(\frac{2y+1}{2y-1}\right) = \ln\left(1+\frac{1}{2y}\right) - \ln\left(1-\frac{1}{2y}\right) = \left(\frac{1}{2y} - \frac{1}{2(2y)^2} + \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} + \frac{1}{5(2y)^5} - \ldots\right)$$

$$- \left(-\frac{1}{2y} - \frac{1}{2(2y)^2} - \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} - \frac{1}{5(2y)^5} - \ldots\right)$$

$$= 2\left(\frac{1}{2y} + \frac{1}{3(2y)^3} + \frac{1}{5(2y)^5} + \ldots\right) > \frac{1}{y}$$ (since all terms after the first are positive).

Again, note that we should justify that the series is valid for $0 < \dfrac{1}{2y} < 1$ i.e. $y > \frac{1}{2}$ in order to justify the use of the given series. It then follows that $\ln\left(\dfrac{2y+1}{2y-1}\right)^y > 1$, and setting $y = n + \frac{1}{2}$ (the crucial final step)

gives $\ln\left(\dfrac{2n+2}{2n}\right)^{n+\frac{1}{2}} > 1 \Rightarrow \left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} > e$.

(iii) This final part only required a fairly informal argument, but the details still required a little bit of care in order to avoid being too vague.

As $n \to \infty$, $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} = \left(1+\dfrac{1}{n}\right)^n \times \left(1+\dfrac{1}{n}\right)^{\frac{1}{2}} \to \left(1+\dfrac{1}{n}\right)^n \times 1+ \to \left(1+\dfrac{1}{n}\right)^n$ from above and e is squeezed into the same limit from both above and below.