

STEP II, 2012 Q4

4 In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n} + \dots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k ,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}.$$

Hence show that $\ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1 + \frac{1}{n}\right)^n < e.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n ,

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}.$$

(iii) Use parts (i) and (ii) to show that as $n \rightarrow \infty$

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e.$$



NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to

NextStepMaths.com